Sequences and Series

Toolkit

Arithmetic Sequences

Arithmetic sequences are sequences with a common *difference*, that is to say, that the difference between consecutive terms is constant.

Description	*Formula*
General k^{th} term	$t_k = a + (k-1)d$, where a is the first term and d is the common difference
Sum of n terms	$S_n = \frac{n}{2}(a + t_n) = \frac{n}{2}(2a + (n-1)d)$
Spacing of terms	Because there is a common difference between consecutive terms we have $t_k + t_l = t_m + t_n$ if and only if $k + l = m + n$

Geometric Sequences

Geometric sequences are sequences with a common *ratio*, that is to say, the ratio of consecutive terms is constant.

Description	*Formula*
General k^{th} term	$t_k = ar^{k-1}$, where a is the first term and r is the common ratio
Sum of n terms	$S_n = \frac{a(1-r^n)}{(1-r)}$
Spacing of terms	If $a \neq 0$ and $r \neq 0$, then because there is a common ratio between consecutive terms, we have $t_k t_l = t_m t_n$ if and only if $k + l = m + n$.
Infinite sum	If the ratio r satisfies the condition $\lvert r \rvert < 1$, we can calculate the infinite sum $a + ar + ar^2 + ar^3 + \cdots$ using $S = \frac{a}{1-r}$.

Other

Arithmetic and geometric sequences are a small subset of all sequences, even though they are emphasized in high school mathematics. The following are some extensions that frequently appear on contests.

Description	*Formula*
Sum of the first n integers	$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$
Sum of the first n squares	$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$
Sum of the first n cubes	$\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$
Telescoping series	If $t_k = u_k - u_{k-1}$, then $\sum_{k=1}^{n} t_k = \sum_{k=1}^{n} (u_k - u_{k-1}) = u_n - u_0$

- A *recursive sequence* is a sequence in which each term can be defined in relation to the previous term (or multiple previous terms).
  - A *recursion formula* defines how to calculate each term t_n from the previous term (or multiple previous terms).
  - Every recursion formula needs to include at least one known or given term. This term is often the initial term, or t_1.
  - The *general term* of a sequence is an expression that is used to calculate each term, t_n, in a sequence directly from its term number, n. This representation of the sequence is sometimes called the *closed form*.

Sample Problems

1. What is the sum of all multiples of 7 or 11 less than 1000?

   Solution

   Since we are adding $(7+14+21+28+\cdots+994)+(11+22+33+\cdots+990)$, we are adding two arithmetic sequences. However the multiples of 77 are included in both sequences and so must be subtracted (in order to avoid counting them twice) after we add the two sequences above. Therefore, the required sum is

$$(7+14+21+28+\cdots+994)+(11+22+33+\cdots+990)-(77+154+\cdots+924).$$

   The term 994 is the 142nd term in the first sequence and so the sum of the first sequence is $\frac{142}{2}(7+994)$. The term 990 is the 90th term in the second sequence and so the sum of the second sequence is $\frac{90}{2}(11+990)$. The term 924 is the 12th term in the sequence of terms we remove and so the sum of that sequence is $\frac{12}{2}(77+924)$. Thus, the required sum is

$$\begin{aligned}&\frac{142}{2}(7+994)+\frac{90}{2}(11+990)-\frac{12}{2}(77+924)\\&=(71+45-6)(1001)\\&=(110)(1001)\\&=110110\end{aligned}$$

2. A sequence is given such that $t_1=1$ and $t_{n+1}=t_n+3n^2+3n+1$. Evaluate t_{100}.

   Solution

   Since the difference, t_n-t_{n-1} is not constant, the series is not arithmetic. Setting $n=1$, we find

$$t_2=1+3+3+1=8$$

   Setting $n=2$, we find

$$t_3=8+12+6+1=27$$

   These facts suggest $t_n=n^3$ for every n.

   To prove that $t_n=n^3$ is an alternate definition for the same sequence, we first note that $t_1=1=1^3$. Further, consider two adjacent terms in the sequence given by the alternate definition, i.e. $t_n=n^3$ and $t_{n+1}=(n+1)^3$. Then the difference between these terms is

$$\begin{aligned}t_{n+1}-t_n&=(n+1)^3-(n)^3\\&=(n^3+3n^2+3n+1)-n^3\\&=3n^2+3n+1\\t_{n+1}&=t_n+3n^2+3n+1\end{aligned}$$

   which matches the original definition of the sequence. We have proved that the original sequence can be expressed as $t_n=n^3$, and thus, $t_{100}=100^3$.

3. If a, b, $a+b$, and ab are positive numbers that form 4 consecutive terms in a geometric sequence, find a.

Solution

Since we have a geometric sequence, the ratios of consecutive terms will be equal . So

$$\frac{a}{b} = \frac{b}{a+b} = \frac{a+b}{ab} \quad (*)$$

Therefore,

$$\begin{aligned} \frac{a}{b} &= \frac{b}{a+b} \\ a^2 + ab &= b^2 \\ b^2 - ab - a^2 &= 0 \\ \left(\frac{b}{a}\right)^2 - \left(\frac{b}{a}\right) - 1 &= 0 \quad (\text{since } a \neq 0) \\ \frac{b}{a} &= \frac{1+\sqrt{5}}{2} \end{aligned}$$

where we have chosen the positive root since a and b are positive. Also from (*),

$$\begin{aligned} \frac{a}{b} &= \frac{a+b}{ab} \\ a^2 &= a+b \\ a &= 1 + \frac{b}{a} \quad (\text{since } a \neq 0) \\ &= \frac{3+\sqrt{5}}{2} \quad (\text{substituting from above}) \end{aligned}$$

Problem Set

1. In a geometric series, $t_5 + t_7 = 1500$ and $t_{11} + t_{13} = 187500$. Find all possible values for the first three terms.

2. Given that a, b and c are consecutive terms in an arithmetic sequence that has distinct terms, calculate x if

$$(b-c)x^2 + (c-a)x + (a-b) = 0$$

3. If x, 4, y are consecutive terms in an arithmetic sequence and x, 3, y are consecutive terms in a geometric sequence, calculate $\frac{1}{x} + \frac{1}{y}$.

4. Three different numbers, whose product is 125, are 3 consecutive terms in a geometric sequence. At the same time they are the first, third and sixth terms of an arithmetic sequence. Find these three numbers.

5. The kth triangular number is given by $T_k = 1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2} = \frac{k^2+k}{2}$. The first six triangular numbers are 1, 3, 6, 10, 15, and 21. Find the sum of the first 200 triangular numbers.

6. If the interior angles of a pentagon form an arithmetic sequence and one interior angle is 90^o, find all possible values of the largest angle in the pentagon.

7. Find the four integers a, b, c and d that satisfy the following conditions:
   - the sum of b and c is 30
   - the sum of a and d is 35
   - the numbers $a < b < c < d$ are in geometric sequence

8. A sequence t_1, t_2, t_3 is formed by choosing t_1 at random from the set $\{1, 2, 3\}$, t_2 at random from the set $\{4, 5, 6\}$, and t_3 at random from the set $\{7, 8, 9\}$. What is the probability that t_1, t_2, t_3 is an arithmetic sequence?

9. The sum of 25 consecutive integers is 500. Determine the smallest of the 25 integers.

10. What is the number of terms in the arithmetic sequence -1994, -1992, -1990, . . ., 1992, 1994?

11. The sum of the first n terms of a sequence is $S_n = 3^n - 1$, where n is a positive integer.
    (a) If t_n represents the nth term of the sequence, determine t_1, t_2, t_3.
    (b) Prove that $\frac{t_{n+1}}{t_n}$ is constant for all values of n.

12. How many terms in the arithmetic sequence 7, 14, 21, . . . are between 40 and 28 001?

13. If f is a function such that $f(1) = 2$ and $f(n+1) = \frac{3f(n)+1}{3}$ for $n = 1, 2, 3, \ldots$, what is the value of $f(100)$?

14. Consider the family of lines with equations of the form $px + qy = r$, and which all pass through the point $(-1, 2)$. Prove that p, q, and r are consecutive terms of an arithmetic sequence.

15. An arithmetic sequence S has terms $t_1, t_2, t_3, \ldots$, where $t_1 = a$ and the common difference is d. The terms t_5, t_9, and t_{16} form a three-term geometric sequence with common ratio r. Prove that S contains an infinite number of three-term geometric sequences, all having the same common ratio r.

16. In the sequence $5, 3, -2, -5, \ldots$, each term after the first two is constructed by taking the preceding term and subtracting the term before it. What is the sum of the first 32 terms in the sequence?

17. Consider the sequence $t_1 = 1, t_2 = -1$ and $t_n = \left(\dfrac{n-3}{n-1}\right) t_{n-2}$ where $n \geq 3$. What is the value of t_{1998}?

18. The nth term of an arithmetic sequence is given by $t_n = 555 - 7n$. If $S_n = t_1 + t_2 + \ldots + t_n$, determine the smallest value of n for which $S_n < 0$.