Grade 11/12 Math Circles

Dynamical Systems and Fractals - Solutions

Exercise Solutions

Exercise 1

Find all of the fixed points of the function $f(x) = x^2 - 2$.

Exercise 1 Solution

Set $f(\bar{x}) = \bar{x}$ and solve.

$$\begin{aligned} \rightarrow \bar{x}^2 - 2 &= \bar{x} \\ \bar{x}^2 - \bar{x} - 2 &= 0 \\ (\bar{x}+1)(\bar{x}-2) &= 0 \end{aligned}$$

This has two solutions: $\bar{x}_1 = -1$ and $\bar{x}_2 = 2$, both of which must be fixed points of $f(x) = x^2 - 2$.

Let's check!

$f(-1) = (-1)^2 - 2 = 1 - 2 = -1$

$f(2) = 2^2 - 2 = 4 - 2 = 2$

Exercise 2

Given that $f(x) = \frac{1}{x}$, find the periodic points of period two of $f(x)$.

Hint: You may want to find the fixed points of $f(x)$ first.

Exercise 2 Solution

First, find the fixed points of $f(x)$ by solving $f(\bar{x}) = \bar{x}$.

$$\frac{1}{\bar{x}} = \bar{x}$$
$$\bar{x}^2 = 1$$

This has solutions $\bar{x}_1 = -1$ and $\bar{x}_2 = 1$. so these must be our fixed points.

Now we want to solve for the fixed points of $f^{[2]}(x)$.

$$\begin{aligned} f^{[2]}(x) &= f(f(x)) \\ &= \frac{1}{1/x} \\ &= x \end{aligned}$$

Setting $f^{[2]}(\bar{x}) = \bar{x}$ gives $\bar{x} = \bar{x}$. This is true for all values of $\bar{x}$. Does this mean that all values of $\bar{x}$ are periodic points of period two of $f(x)$? Almost!

Since the points $\bar{x}_1 = -1$ and $\bar{x}_2 = 1$ are fixed points of $f(x)$, they cannot also be periodic points of period two. We also need to be careful and consider the domain of $f(x)$, which excludes the point $x = 0$ (since $f(x) = \frac{1}{x}$ is undefined when $x = 0$). This means that $x = 0$ cannot be a periodic point of $f(x)$. What we are left with is that all $x \in \mathbb{R}$ except for $x = -1, 1$, and 0 are periodic points of period two of $f(x)$. We could write the set of periodic points of period two of $f(x)$ as $\{x \in \mathbb{R} | x \neq -1, 1, 0\}$.

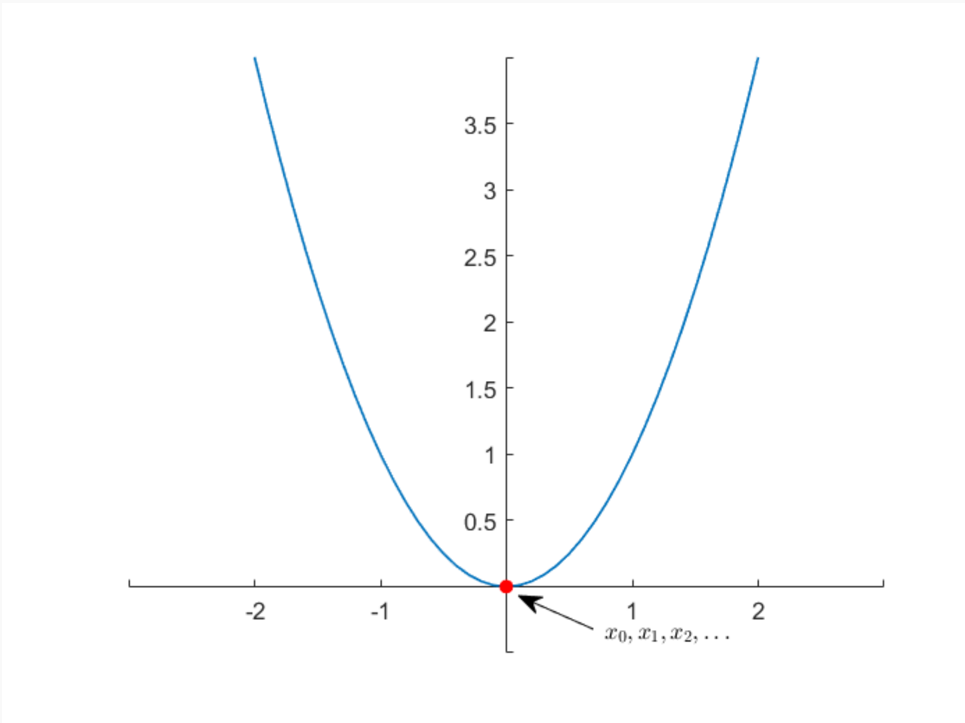
Problem Set Solutions

1. Consider the function $f(x) = x^2$. Sketch this function and plot the first few points of its orbit $\{x_0, x_1, x_2, x_3, \ldots\}$, i.e. plot the points $(x_0, x_1 = f(x_0))$, $(x_1, x_2 = f(x_1))$, etc..., for the starting values $x_0 = 0, 1/2$, and 2. Describe what is happening to the orbit of $f(x)$ for each of these starting values.

Solution:

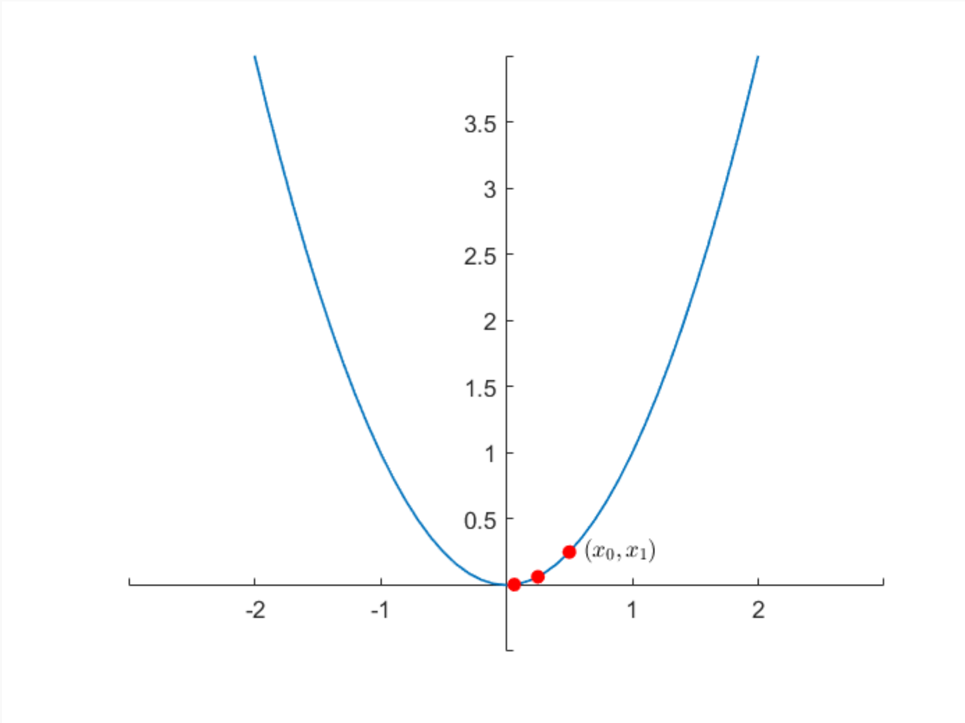
$x_0 = 0$:

The orbit of $x_0 = 0$ under $f(x) = x^2$ is: $\{0, 0, 0, 0, \ldots\}$. The point $x_0 = 0$ is a fixed point of $f(x)$.



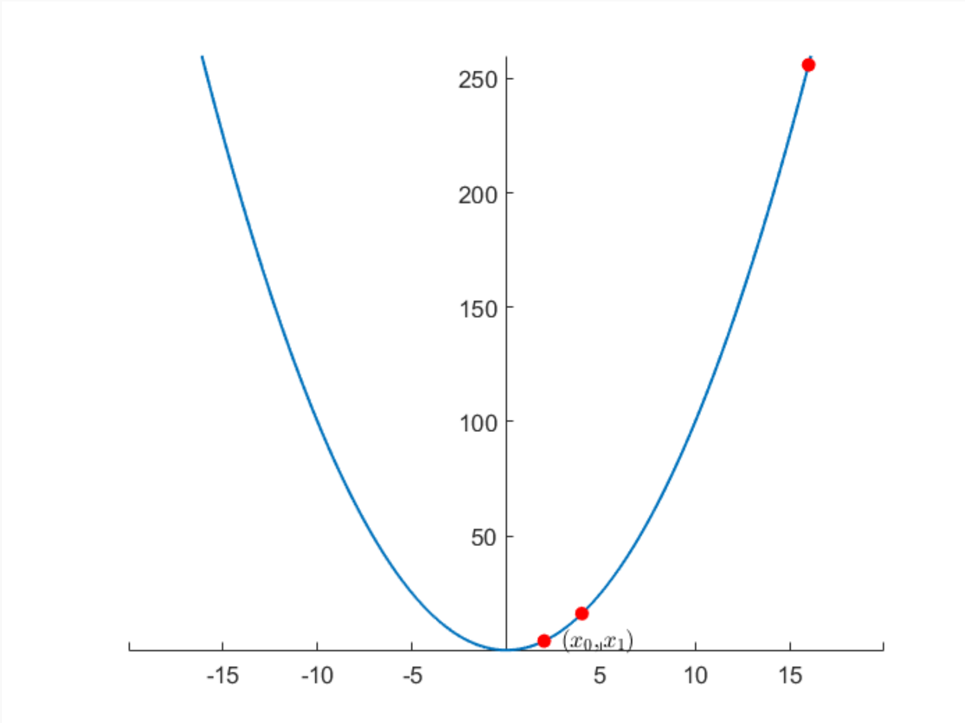
$x_0 = 1/2$:

The orbit of $x_0 = 1/2$ under $f(x) = x^2$ is: $\{1/2, 1/4, 1/16, 1/256, \ldots\}$. The iterates of $x_0 = 1/2$ are getting smaller in magnitude on each iteration and approaching zero.



$x_0 = 2$:

The orbit of $x_0 = 2$ under $f(x) = x^2$ is: $\{2, 4, 16, 256, \ldots\}$. The iterates of $x_0 = 2$ are getting larger in magnitude on each iteration and approaching infinity.



2. Let $f(x) = x^2 + 3x + 1$. Find all of the fixed points of $f(x)$.

Solution: To find the fixed points we need to solve $f(\bar{x}) = \bar{x}$.

$$\begin{aligned} \bar{x}^2 + 3\bar{x} + 1 &= \bar{x} \\ \bar{x}^2 + 2\bar{x} + 1 &= 0 \\ (\bar{x} + 1)^2 &= 0 \end{aligned}$$

This has one solution, $\bar{x} = -1$, so $f(x)$ has just one fixed point at $\bar{x} = -1$.

3. Consider the family of functions defined by $f_c(x) = cx$ where c is a constant and $c \neq 0$. Determine all of the fixed points of $f_c(x)$.

Hint: You may end up with different fixed points depending on the value of c.

Solution: To find the fixed points of $f_c(x)$ we solve $f_c(\bar{x}) = \bar{x}$, treating c as a constant.

$$c\bar{x} = \bar{x}$$
$$(c-1)\bar{x} = 0$$

For most values of c, this has one solution, $\bar{x} = 0$, however we can see that when $c = 1$, then all $\bar{x} \in \mathbb{R}$ are solutions. Thus, the fixed points of $f_c(x)$ are $\bar{x} = 0$, $c \neq 1$ and $\bar{x} \in \mathbb{R}$, $c = 1$. Notice that when $c = 1$, $f_1(x) = x$, for which all $x \in \mathbb{R}$ are clearly fixed points.

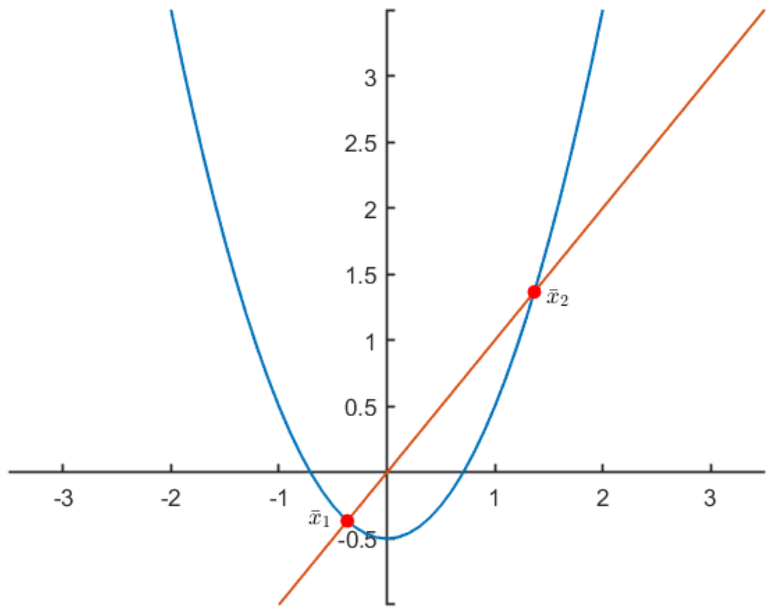
4. (a) Consider the function $f(x) = x^2 - \frac{1}{2}$. Sketch $f(x)$ and $y = x$ on the same set of axes and show graphically that $f(x)$ has two fixed points. Label these fixed points on your sketch as $\bar{x}_1$ and $\bar{x}_2$ such that $\bar{x}_1 < \bar{x}_2$.

   (b) Use a graphical method (i.e. cobweb diagram) to help determine the behaviour of various orbits starting near both $\bar{x}_1$ and $\bar{x}_2$. Use your diagram to make an educated guess as to the nature (attractive, repelling, or neither) of each fixed point.

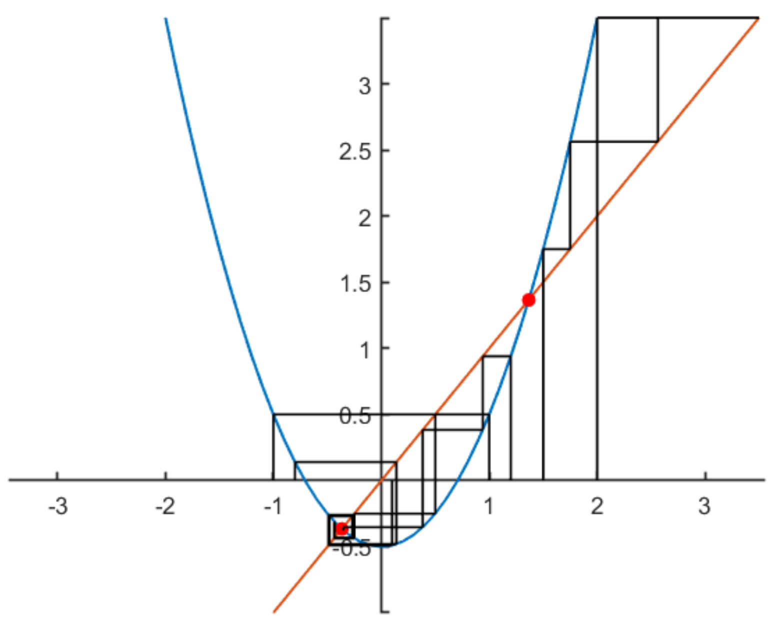
   (c) Now consider the family of functions $f_c(x) = x^2 + c$ where c is a constant. For what values of c do fixed points of $f_c(x)$ exist? Some sketches of the graphs of $f_c(x)$ for various values of c may help, but they are not necessary.

Solution:

(a) From the graph, we can see that $f(x)$ intersects the line $y = x$ twice, and thus has two fixed points.



(b) Cobweb diagram:



From our cobweb diagram we see that the iterates of $f(x)$ are attracted towards the fixed point $\bar{x}_1$, so we can guess that this is an attractive fixed point. On the other hand, the iterates of $f(x)$ move away from the fixed point $\bar{x}_2$, so this is likely to be a repelling fixed point.

(c) To find the fixed points of $f_c(x)$ we need to solve $f_c(\bar{x}) = \bar{x}$.

$$\begin{aligned} \bar{x}^2 + c &= \bar{x} \\ \bar{x}^2 - \bar{x} + c &= 0 \end{aligned}$$

Using the quadratic formula, this has solutions $\bar{x} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4c}$. This has (real) solutions only when the argument of the square root is greater than (or equal to) zero, i.e.

$$\begin{aligned} 1 - 4c &\geq 0 \\ c &\leq \frac{1}{4}. \end{aligned}$$

Thus, $f_c(x)$ has fixed points when $c \leq \frac{1}{4}$. Notice that when $c = \frac{1}{4}$, $f_c(x)$ has one fixed point (only one solution to the quadratic formula) and when $c < \frac{1}{4}$, $f_c(x)$ has two fixed points (two solutions to the quadratic formula).

5. Let $f(x) = -x^3$. Find all fixed points and periodic points of period two of $f(x)$.

Solution: First, let's find any fixed points by solving $f(\bar{x}) = \bar{x}$.

$$\begin{aligned} -\bar{x}^3 &= \bar{x} \\ \bar{x} + \bar{x}^3 &= 0 \\ \bar{x}(1 + \bar{x}^2) &= 0 \end{aligned}$$

This has just one solution, $\bar{x} = 0$, so $f(x)$ has one fixed point at $\bar{x} = 0$.

Now, let's find any periodic points of period two. We need to solve $f^{[2]}(\bar{x}) = \bar{x}$.

$$\begin{aligned} f^{[2]}(x) &= f(f(x)) \\ &= -\left(-x^3\right)^3 \\ &= x^9 \end{aligned}$$

So we need to solve $\bar{x}^9 = \bar{x}$. Rearranging, this gives $\bar{x}(\bar{x}^8 - 1) = 0$, which has three solutions $\bar{x} = 0, -1$, and 1. Since $\bar{x} = 0$ is a fixed point, the remaining two points must form a two cycle. Thus, $\bar{x} = -1$ and 1 are the points of period two of $f(x)$.

6. **CHALLENGE** Let $f(x) = 1 - x^2$. Find all fixed points and periodic points of period two of $f(x)$.

Solution: First, let's find the fixed points.

$$\begin{aligned} 1 - \bar{x}^2 &= \bar{x} \\ \bar{x}^2 + \bar{x} - 1 &= 0 \end{aligned}$$

Using the quadratic formula, this gives two solutions, $\bar{x} = \frac{-1}{2} \pm \frac{\sqrt{5}}{2}$, which are the two fixed points of $f(x)$.

Next, we want to solve for any periodic points of period two. First we find $f^{[2]}(x)$,

$$\begin{aligned} f^{[2]}(x) &= 1 - \left(1 - x^2\right)^2 \\ &= 1 - 1 + 2x^2 - x^4 \\ &= 2x^2 - x^4 \end{aligned}$$

and then solve $f^{[2]}(\bar{x}) = \bar{x}$.

$$\begin{aligned} 2\bar{x}^2 - \bar{x}^4 &= \bar{x} \\ \bar{x}^4 - 2\bar{x}^2 + \bar{x} &= 0 \\ \bar{x}\left(\bar{x}^3 - 2\bar{x} + 1\right) &= 0 \end{aligned}$$

Factoring the cubic part of this expression could be difficult, but luckily we know that the fixed points of $f(x)$ are also solutions to $f^{[2]}(\bar{x}) = \bar{x}$. This means that $(\bar{x}^2 + \bar{x} - 1)$ must be a factor. Thus we have

$$\bar{x}\left(\bar{x}^2 + \bar{x} - 1\right)(\bar{x} - 1) = 0$$

The two new solutions (which are not fixed points of the original function) are $\bar{x} = 0$ and 1. Thus we must have the two cycle $\{0, 1\}$.

7. **CHALLENGE** Consider the function $f(x) = x + \cos(x)$. Show that $f(x)$ has an infinite number of fixed points.

Solution: To find the fixed points of $f(x)$ we must solve

$$\begin{aligned} \bar{x} + \cos(\bar{x}) &= \bar{x} \\ \cos(\bar{x}) &= 0. \end{aligned}$$

Considering the unit circle (or a graph of $\cos(x)$), we see that this is true when $\bar{x}$ is an odd multiple of 90 degrees, i.e. $\bar{x} = 90°, 270°, 450°, -90°, \ldots$

We can write this as $\bar{x}$ is a fixed point of $f(x)$ when $\bar{x} = (2k + 1) \cdot 90°$ for any integer k. Since the set of all integers is an infinite set, this results in an infinite number of fixed points for $f(x)$.