Problem of the Week

Problem E and Solution

The Choice is Yours

Problem

You are given the following list of the seven numbers: 10, 2, 5, 2, 4, 6, 2. You are asked to choose a positive number as the eighth number so that

- the mean, median, and mode of the eight numbers are distinct, and
- the mean, median, and mode differ by the same amount when put in order from least to greatest.

Determine all possible choices for the eighth number.

Solution

Let x represent the number chosen.

Since there are at least three 2s, the mode will be 2 regardless the value of x.

The mean of the numbers is $\frac{10+2+5+2+6+4+2+x}{8} = \frac{x+31}{8}$.

The median of the numbers will depend on the value of x compared to the other numbers. Since there are eight numbers in the list, the median will be the average of the fourth and fifth numbers in the list, ordered from least to greatest. We will break the problem into cases.

Case 1: $0 < x \leq 2$

Then the list, ordered from least to greatest, is x, 2, 2, 2, 4, 5, 6, 10.

The median is the average of the fourth and fifth numbers, or $\frac{2+4}{2} = 3$.

Since $x > 0$, the mean is $\frac{x+31}{8} > \frac{31}{8} > 3$. It follows that the mean is greater than the median. Then the mean, median, and mode in order from least to greatest is 2, 3, $\frac{x+31}{8}$.

Since the differences between adjacent numbers are equal, we have

$$\begin{aligned} 3 - 2 &= \frac{x+31}{8} - 3 \\ 4 &= \frac{x+31}{8} \\ 32 &= x + 31 \\ 1 &= x \end{aligned}$$

When $x = 1$, the list of numbers, ordered from least to greatest, is 1, 2, 2, 2, 4, 5, 6, 10. The mean is 4, the median is 3, and the mode is 2. When listed from least to greatest, the three numbers are 2, 3, 4 and the difference between adjacent terms is 1. Therefore, $x = 1$ is a possible choice for the eighth number.

Case 2: $2 < x \leq 5$

Then the list, ordered from least to greatest, is 2, 2, 2, x, 4, 5, 6, 10 or 2, 2, 2, 4, x, 5, 6, 10.

In both lists, the fourth and fifth numbers are x and 4. It follows that the median for both lists is $\frac{x+4}{2}$. We also know that the mode is 2. However, we do not know which is larger, the median or the mean, so we examine both cases.

- **Case 2a:** The median is less than the mean.

  In this case, the mean, median, and mode, ordered from least to greatest, is 2, $\frac{x+4}{2}$, $\frac{x+31}{8}$. Since the differences between adjacent numbers are equal, we have

$$\frac{x+4}{2} - 2 = \frac{x+31}{8} - \frac{x+4}{2}$$

  Multiplying both sides of the equation by 8,

$$\begin{aligned} 4x + 16 - 16 &= x + 31 - 4x - 16 \\ 7x &= 15 \\ x &= \frac{15}{7} \end{aligned}$$

  When $x = \frac{15}{7}$, the list of numbers, ordered from least to greatest, is 2, 2, 2, $\frac{15}{7}$, 4, 5, 6, 10. The mean is $\frac{29}{7}$, the median is $\frac{43}{14}$, and the mode is 2. When listed from least to greatest, the three numbers are 2, $\frac{43}{14}$, $\frac{29}{7}$, and the difference between adjacent terms is $\frac{15}{14}$. Therefore, $x = \frac{15}{7}$ is a possible choice for the eighth number.

- **Case 2b:** The median is greater than the mean.

  In this case, the mean, median, and mode, ordered from least to greatest, is 2, $\frac{x+31}{8}$, $\frac{x+4}{2}$. Since the differences between adjacent numbers are equal, we have

$$\frac{x+31}{8} - 2 = \frac{x+4}{2} - \frac{x+31}{8}$$

  Multiplying both sides of the equation by 8,

$$\begin{aligned} x + 31 - 16 &= 4x + 16 - x - 31 \\ 30 &= 2x \\ x &= 15 \end{aligned}$$

  But $15 > 5$, so there is no value of x that satisfies the given conditions in this case.

Case 3: $x > 5$

Then the first five numbers in the list, in order from least to greatest, are 2, 2, 2, 4, 5.

Since the fourth number in the list is 4 and the fifth number is 5, then the median is $\frac{9}{2}$. The mode is 2. Since $x > 5$, the mean is $\frac{x+31}{8} > \frac{36}{8} = \frac{9}{2}$. It follows that the mean is greater than the median. The mean, median, and mode, ordered from least to greatest, is then 2, $\frac{9}{2}$, $\frac{x+31}{8}$.

Since the differences between adjacent numbers in the list are equal, we have

$$\begin{aligned} \frac{9}{2} - 2 &= \frac{x+31}{8} - \frac{9}{2} \\ 7 &= \frac{x+31}{8} \\ 56 &= x + 31 \\ 25 &= x \end{aligned}$$

When $x = 25$, the list of numbers, ordered from least to greatest, is 2, 2, 2, 4, 5, 6, 10, 25. The mean is 7, the median is $\frac{9}{2}$, and the mode is 2. When listed from least to greatest, the three numbers are 2, $\frac{9}{2}$, 7 and the difference between adjacent terms is $\frac{5}{2}$. Therefore, $x = 25$ is a possible choice for the eighth number.

Therefore, the possible choices for the eighth number are 1, $\frac{15}{7}$ and 25.