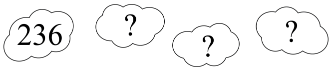
Problem of the Week



Problem E and Solution

Four Numbers

Problem

Norbert has four favourite numbers. Each of these is a three-digit number ABC with the following two properties:

1. The digits A, B, and C are all different.
2. The product $A \times B \times C$ is equal to the two-digit number BC.

For example, one of Norbert's favourite numbers is 236, since $2 \times 3 \times 6 = 36$.

Find Norbert's other three favourite numbers.

NOTE: It may be helpful to recall that any two-digit number of the form BC can be represented by the sum $10B + C$. For example, $32 = 10(3) + 2$.

Solution

We know that there are only four answers, so we could attempt a trial and error approach to find the remaining three numbers, or even write a program that could find them using a brute force approach. However here we will present a more systematic approach.

First, since A, B, and C are digits, they must be positive integers between 0 and 9, inclusive. Also, since the product $A \times B \times C$ is a two-digit number, none of A, B, or C can equal zero.

We want to find all three-digit numbers ABC such that $A \times B \times C = 10B + C$. Since $B \neq 0$, we can divide by B and the problem becomes equivalent to finding all integers A, B, C with $1 \leq A, B, C \leq 9$ and A, B, and C distinct such that

$$A \times C = 10 + \frac{C}{B}$$

Since A and C are integers, then so is $A \times C$, so it follows that $\frac{C}{B}$ must be an integer as well. Therefore, for each possible value of C, we must have that B divides exactly into C.

We will break the problem into cases based on the value of C, and then sub-cases based on the possible values of B.

- **Case 1:** $C = 1$
  There are no values of B where $\frac{C}{B}$ is an integer and $B \neq C$.
- **Case 2:** $C = 2$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 2$, $A \times B \times C = 10B + C$ becomes $2A = 12$ and so $A = 6$. Therefore, one of the three-digit numbers is 612.
- **Case 3:** $C = 3$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 3$, $A \times B \times C = 10B + C$ becomes $3A = 13$ and so $A = \frac{13}{3}$. Since A is not an integer, there is no solution in this case.

- **Case 4:** $C = 4$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$ or $B = 2$.
  - **Case 4a:** $B = 1$
    In this case, $A \times B \times C = 10B + C$ becomes $4A = 14$ and so $A = \frac{7}{2}$. Since A is not an integer, there is no solution in this case.
  - **Case 4b:** $B = 2$
    In this case, $A \times B \times C = 10B + C$ becomes $8A = 24$ and so $A = 3$. Therefore, one of the three-digit numbers is 324.
- **Case 5:** $C = 5$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$. If $B = 1$ and $C = 5$, $A \times B \times C = 10B + C$ becomes $5A = 15$ and so $A = 3$. Therefore, one of the three-digit numbers is 315.
- **Case 6:** $C = 6$
  For $\frac{C}{B}$ to be an integer and $B \neq C$, we must have $B = 1$, $B = 2$, or $B = 3$.
  - **Case 6a:** $B = 1$
    In this case, $A \times B \times C = 10B + C$ becomes $6A = 16$ and so $A = \frac{8}{3}$. Since A is not an integer, there is no solution in this case.
  - **Case 6b:** $B = 2$
    In this case, $A \times B \times C = 10B + C$ becomes $12A = 26$ and so $A = \frac{13}{6}$. Since A is not an integer, there is no solution in this case.
  - **Case 6c:** $B = 3$
    In this case, $A \times B \times C = 10B + C$ becomes $18A = 36$ and so $A = 2$. Therefore, one of the three-digit numbers is 236. This is the number given in the example.

We can actually stop here since we have found 4 different three-digit numbers that satisfy the conditions outlined in the problem. If we had not been given the number of possible solutions, we would need to continue by checking cases when $C = 7$, $C = 8$, and $C = 9$. It turns out that there are no solutions in these cases.

Therefore, Norbert's four favourite numbers are 612, 324, 315, and 236.