$$\frac{A}{B}-\frac{B}{A}=\frac{A+B}{AB}$$

Problem of the Week
Problem E and Solution
Fraction Distraction

Problem

Determine the number of solutions to the equation

$$\frac{A}{B}-\frac{B}{A}=\frac{A+B}{AB}$$

where A and B are both integers, $-9 \leq A \leq 9$, and $-9 \leq B \leq 9$.

Solution

First notice that neither A nor B can equal zero. Starting with the equation, we simplify as follows.

$$\begin{aligned}
\frac{A}{B}-\frac{B}{A} &= \frac{A+B}{AB}\\
\frac{A^2}{AB}-\frac{B^2}{AB} &= \frac{A+B}{AB}\\
\frac{A^2-B^2}{AB} &= \frac{A+B}{AB}\\
\frac{(A-B)(A+B)}{AB} &= \frac{A+B}{AB}\\
(A-B)(A+B) &= A+B
\end{aligned}$$

Since the two sides are equal, $A-B=1$ or $A+B=0$. We will consider these two cases.

Case 1: $A-B=1$

In this case, we know that A and B differ by 1 and $A>B$. The largest value A can be is 9. When $A=9$, $B=8$. The smallest value B can be is -9. When $B=-9$, $A=-8$, a value which is 1 more than the value of B.

So A can take on all of the integer values from -8 to 9, except $A=0$. But when $A=1$, $B=0$, so we have to remove this value of A as well. There are 18 values for A from -8 to 9. After removing $A=0$ and $A=1$, there are 16 values for A and therefore 16 corresponding values for B. Thus, the equation has 16 solutions when $A-B=1$.

Case 2: $A+B=0$

In this case, $A+B=0$ or $A=-B$. The largest value A can be is 9. When $A=9$, $B=-9$. The smallest value A can be is -9. When $A=-9$, $B=9$. So A can take on all of the integer values from -9 to 9, except $A=0$. Thus, the equation has $19-1=18$ solutions when $A+B=0$.

Therefore, there are $16+18=34$ solutions to the equation which satisfy the conditions that A and B are both integers, $-9 \leq A \leq 9$, and $-9 \leq B \leq 9$.