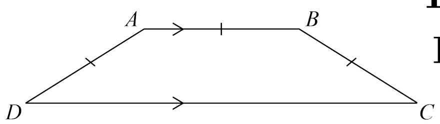


Problem of the Week
Problem E and Solution
Trapped

Problem

In trapezoid $ABCD$, sides AB and CD are parallel, and the lengths of sides AD, AB, and BC are equal. If the perpendicular distance between AB and CD is 8 units, and the length of CD is equal to 4 more than half the sum of the other three side lengths, determine the area and perimeter of trapezoid $ABCD$.

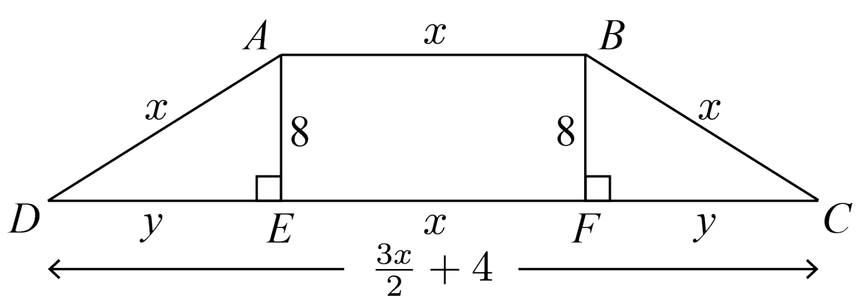
Solution

Let x represent the length of AD. Then $AD = AB = BC = x$. Since CD is equal to 4 more than half the sum of the other three side lengths, $CD = \frac{3x}{2} + 4$. Note that since $x > 0$, $\frac{3}{2}x + 4 > x$ and so CD is longer than AB.

Draw altitudes from A and B meeting CD at E and F, respectively. Then $AE = BF = 8$.

Let y represent the length of DE. We can show that $DE = CF$ using the Pythagorean Theorem as follows: $DE^2 = AD^2 - AE^2 = x^2 - 8^2 = x^2 - 64$ and $CF^2 = BC^2 - BF^2 = x^2 - 8^2 = x^2 - 64$. Then $CF^2 = x^2 - 64 = DE^2$, so $CF = DE = y$ since $CF > 0$.

Since $\angle AEF = \angle BFE = 90^\circ$ and AB is parallel to CD, it follows that $\angle BAE = \angle ABF = 90^\circ$ and $ABFE$ is a rectangle so $AB = EF = x$.



We can now determine the relationship between x and y.

$$\begin{aligned} CD &= DE + EF + CF \\ \frac{3x}{2} + 4 &= y + x + y \\ \frac{x}{2} + 4 &= 2y \\ \frac{x}{4} + 2 &= y \end{aligned}$$

So $DE = CF = \frac{x}{4} + 2$. We then use the Pythagorean Theorem in $\triangle AED$.

$$\begin{aligned} AD^2 &= AE^2 + DE^2 \\ x^2 &= 8^2 + \left(\frac{x}{4} + 2\right)^2 \\ x^2 &= 64 + \frac{x^2}{16} + x + 4 \\ 0 &= \frac{15x^2}{16} - x - 68 \end{aligned}$$

Using the quadratic formula, we can determine the value of x.

$$\begin{aligned} x &= \frac{1 \pm \sqrt{1 - 4\left(\frac{15}{16}\right)(-68)}}{2\left(\frac{15}{16}\right)} \\ &= \frac{1 \pm \sqrt{256}}{\frac{15}{8}} \\ &= \frac{8 \pm 128}{15} \end{aligned}$$

Thus, $x = -8$ or $x = \frac{136}{15}$. Since $x > 0$, we can conclude that $x = \frac{136}{15}$, so $AD = AB = BC = \frac{136}{15}$. Then $CD = \left(\frac{3}{2}\right)\left(\frac{136}{15}\right) + 4 = \frac{88}{5}$. Then we can calculate the area and perimeter of trapezoid $ABCD$.

$$\begin{aligned} \text{Area of } ABCD &= \frac{1}{2} \times AE \times (AB + CD) \\ &= \frac{1}{2} \times 8 \times \left(\frac{88}{5} + \frac{136}{15}\right) \\ &= \frac{320}{3} \end{aligned}$$

$$\begin{aligned} \text{Perimeter of } ABCD &= DA + AB + BC + CD \\ &= 3\left(\frac{136}{15}\right) + \frac{88}{5} \\ &= \frac{224}{5} \end{aligned}$$

Therefore the area of trapezoid $ABCD$ is $\frac{320}{3}$ units^2 and the perimeter is $\frac{224}{5}$ units.