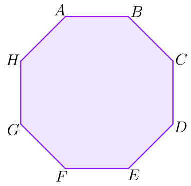


Problem of the Week
Problem E and Solution
Octosquares?

Problem

If the vertices of a regular octagon are randomly labelled A, B, C, D, E, F, G, and H and each letter is used exactly once, what is the probability that $ABCD$ is a square?

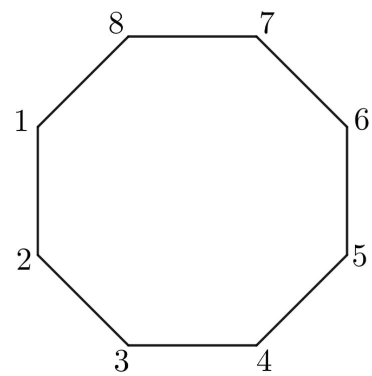
Solution

Solution 1

To determine the probability, we determine the number of ways to label the vertices of the regular octagon so that $ABCD$ forms a square and divide by the total number of ways the regular octagon can be labelled.

First, we determine the total number of ways that the vertices of a regular octagon can be labelled A, B, C, D, E, F, G, and H.

Let's start with the top left vertex. There are 8 possible ways to label it (it can be labelled as A, B, C, D, E, F, G, or H). Moving clockwise, the next vertex can be labelled 7 different ways (it can be assigned any letter other than the letter assigned to the previous vertex). Moving clockwise, the next vertex can be labelled 6 different ways (it can be assigned any letter other than the 2 that have already been used). Moving clockwise, the next vertex can be assigned 5 different ways, and so on. Once we reach the last vertex there will only be 1 letter left, so it can be assigned a letter only 1 way.



Therefore, there are

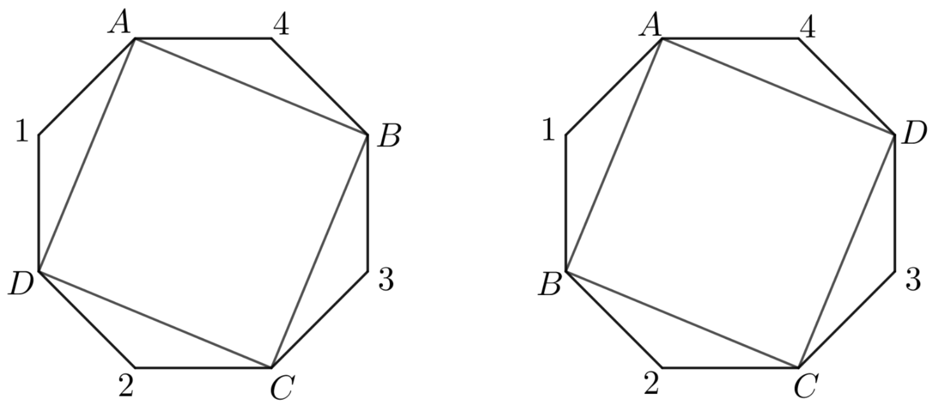
$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = 40\,320$$

different ways to label the regular octagon with the letters A, B, C, D, E, F, G, and H.

Now, let's determine how many of the $40\,320$ labellings result in $ABCD$ forming a square.

Let's suppose vertex A is the top left vertex. Then there are two possible ways to label B, C, and D so that $ABCD$ forms a square.

For each of these two labellings, there are 4 choices for labelling the vertex to the right of A (it can be assigned E, F, G, or H). Given the labelling of that vertex, moving clockwise, there are 3 choices for the next unlabelled vertex, then 2 choices for the next unlabelled vertex, and then 1 choice for the last unlabelled vertex.



Therefore, for each case, there are $4 \times 3 \times 2 \times 1 = 24$ ways to label the remaining vertices. Therefore, there are $24 + 24 = 48$ ways to label the regular octagon with A being the top left vertex and $ABCD$ forming a square.

Using a similar argument, we can see that for any vertex that A can be assigned to, there will be 48 ways to label the regular octagon so that $ABCD$ forms a square. Since A can be assigned to 8 different vertices, there are $8 \times 48 = 384$ different ways to label the regular octagon so that $ABCD$ forms a square.

Therefore, the probability that $ABCD$ forms a square is $\dfrac{384}{40\,320} = \dfrac{1}{105}$.

Solution 2

The first solution counts the number of arrangements where $ABCD$ forms a square, and divides by the total number of possible arrangements. This solution uses a more direct probability argument.

The label A can go anywhere.

There is now two spots where label B can be placed to form a square, so a $\frac{2}{7}$ chance that the B will be placed in a location to form a square.

There is now one spot where C must be placed (across from the A), so a $\frac{1}{6}$ chance that the C will be placed in a location to form a square.

Since there are 5 vertices left to be labelled, there is now a $\frac{1}{5}$ chance that the D will be placed in the only valid location to form a square.

Thus, the probability that $ABCD$ forms a square is

$$\frac{2}{7} \times \frac{1}{6} \times \frac{1}{5} = \frac{2}{210} = \frac{1}{105}$$