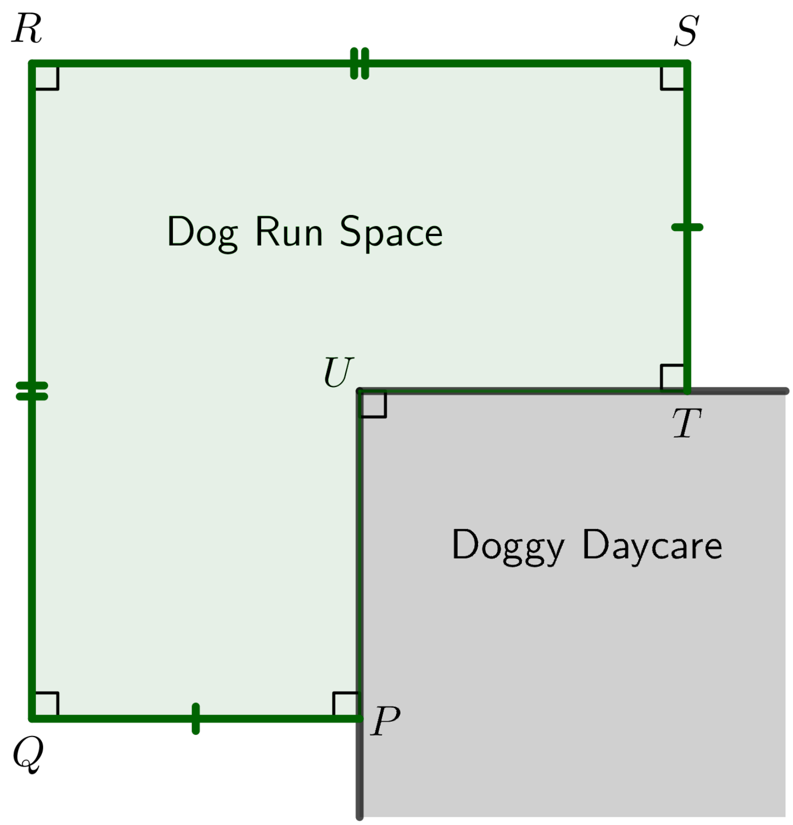
Problem of the Week
Problem E and Solution
Run Dog Run

Problem

At POTW Doggy Care, there is a need for a new outdoor dog run space. The layout of the dog run space is represented by $PQRSTU$ in the diagram below.



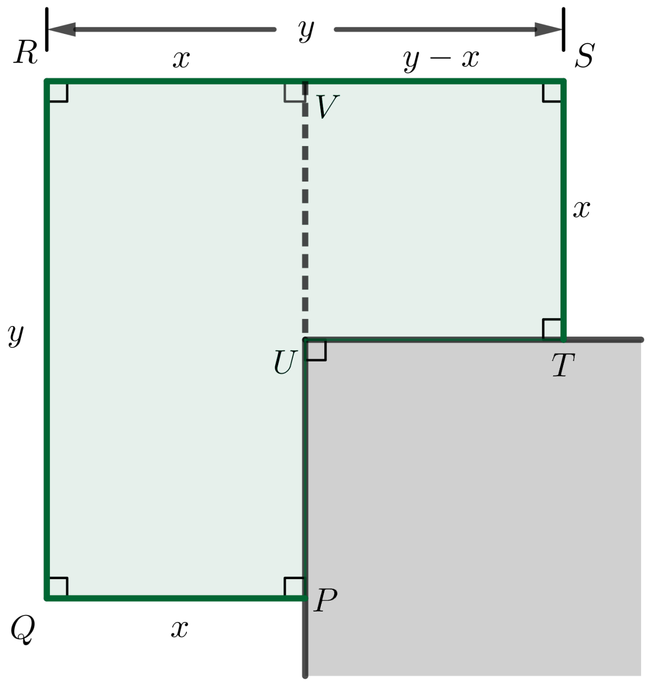
The lengths of the two longer sides, QR and RS, are to be the same, and the lengths of the two shorter sides, PQ and ST, are to be the same. There will be right angles at each corner.

The dog run space is to be built using a fence along PQ, QR, RS, and ST, and using the walls of the daycare along PU and TU. The total fencing to be used is 30 m. Determine the dimensions of the dog run space that will give the maximum area for the dog run.

Solution

Extend PU to RS, letting the intersection point be V. Then $PV \perp RS$.

Let x represent the lengths, in metres, of both PQ and ST. Let y represent the lengths, in metres, of both QR and RS. Since $PQRV$ is a rectangle, $RV = PQ = x$ and $VS = RS - RV = y - x$.



The total length of fencing from P to Q to R to S to T is

$$PQ + QR + RS + ST = x + y + y + x = 2x + 2y$$

Since the total amount of fencing used is 30 m, we have $2x + 2y = 30$. Thus, $x + y = 15$ and $y = 15 - x$.

$$\begin{aligned}\text{Area of dog run} &= \text{Area } PQRV + \text{Area } VSTU \\ &= QR \times RV + VS \times ST \\ &= yx + (y - x)x \\ &= 2xy - x^2\end{aligned}$$

Substituting $y = 15 - x$, this becomes

$$\begin{aligned}\text{Area of dog run} &= 2x(15 - x) - x^2 \\ &= 30x - 2x^2 - x^2 \\ &= -3x^2 + 30x\end{aligned}$$

Completing the square, we have

$$\begin{aligned}\text{Area of dog run} &= -3(x^2 - 10x) \\ &= -3(x^2 - 10x + 5^2 - 5^2) \\ &= -3(x^2 - 10x + 25) + 75 \\ &= -3(x - 5)^2 + 75\end{aligned}$$

This is the equation of a parabola which opens down from a vertex of $(5, 75)$. Thus, the maximum area is 75 m^2, and occurs when $x = 5$ m. When $x = 5$, we have $y = 15 - x = 15 - 5 = 10$ m.

Therefore, if $QR = RS = 10$ m and $PQ = ST = 5$ m, this gives a maximum area of 75 m^2.