Problem of the Week

Problems and Solutions 2023-2024

Problem E

Grade 11/12

The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

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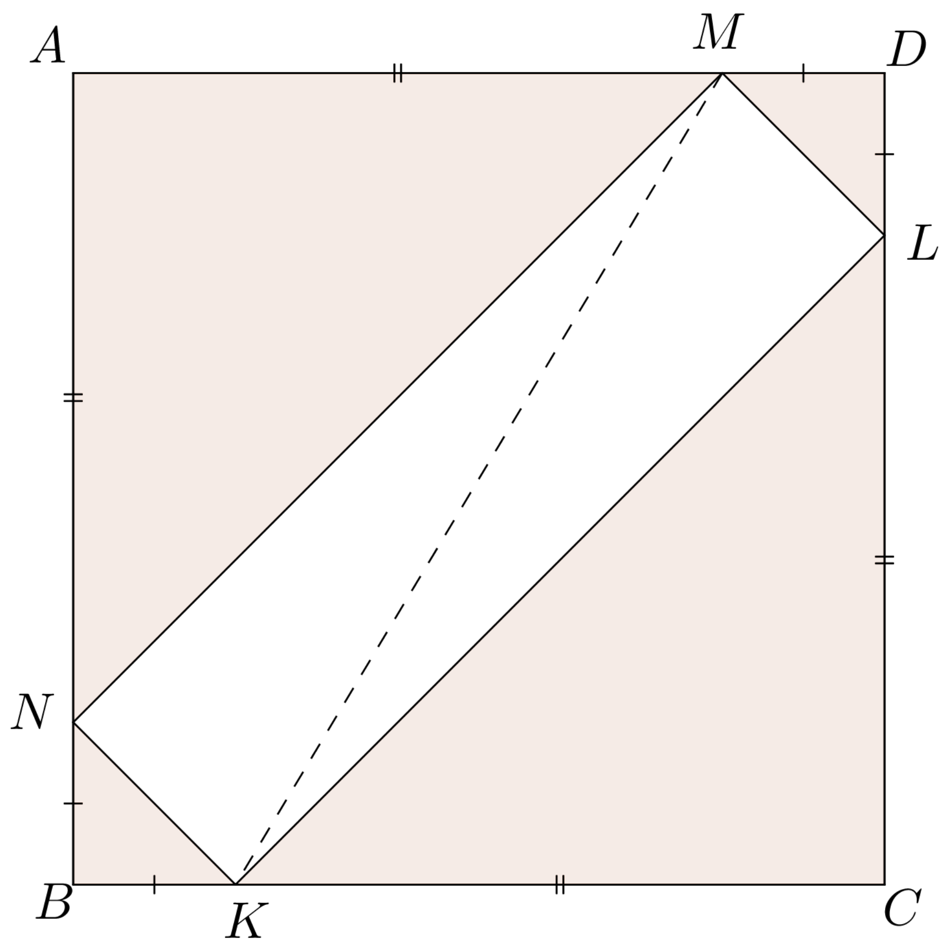
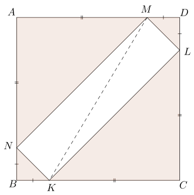
Square $ABCD$ has K on BC, L on DC, M on AD, and N on AB such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm^2, what is the length of MK?





Problem of the Week

Problem E and Solution

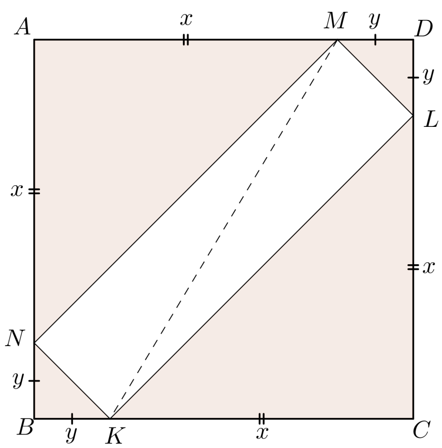
Diagonal Distance

Problem

Square $ABCD$ has K on BC, L on DC, M on AD, and N on AB such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm^2, what is the length of MK?

Solution

Let x represent the lengths of the equal sides of $\triangle AMN$ and $\triangle LKC$, and let y represent the lengths of the equal sides of $\triangle MDL$ and $\triangle BNK$.



Thus, area $\triangle AMN =$ area $\triangle LKC = \frac{1}{2}x^2$, and area $\triangle MDL =$ area $\triangle BNK = \frac{1}{2}y^2$.

Therefore, the total area of the four triangles is equal to $\frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 = x^2 + y^2$. Since we're given that this area is 50 cm^2, we have $x^2 + y^2 = 50$.

Three different solutions to find the length of MK are provided.

Solution 1

In $\triangle AMN$, $MN^2 = AM^2 + AN^2 = x^2 + x^2$, and in $\triangle BNK$, $NK^2 = BN^2 + BK^2 = y^2 + y^2$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= x^2 + x^2 + y^2 + y^2 \\ &= x^2 + y^2 + x^2 + y^2 \\ &= 50 + 50 \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 2

In $\triangle AMN$, $MN^2 = x^2 + x^2 = 2x^2$. Therefore, $MN = \sqrt{2}x$, since $x > 0$.
In $\triangle BNK$, $NK^2 = y^2 + y^2 = 2y^2$. Therefore, $NK = \sqrt{2}y$, since $y > 0$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

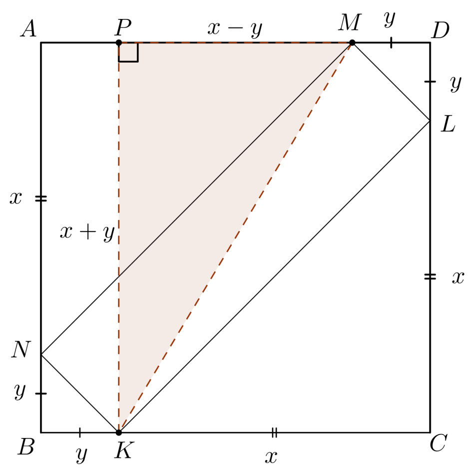
$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= (\sqrt{2}x)^2 + (\sqrt{2}y)^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 3

We construct the line segment KP, where P lies on AD such that KP is perpendicular to AD.

Then $APKB$ is a rectangle. Furthermore, $AP = BK = y$, $PK = AB = x + y$, and
$PM = AM - AP = x - y$.



Since $\triangle PKM$ is a right-angled triangle, by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= PM^2 + PK^2 \\ &= (x - y)^2 + (x + y)^2 \\ &= x^2 - 2xy + y^2 + x^2 + 2xy + y^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Problem of the Week

Problem E

Discarding Digits

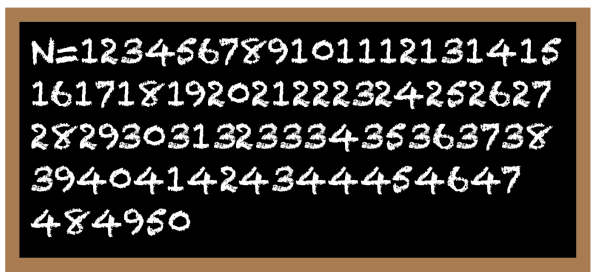
Stef forms the integer N by writing the integers from 1 to 50 in order.

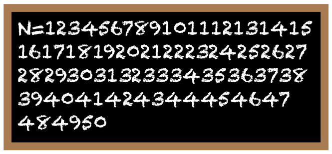
That is,

$N = 1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.$

Stef then selects some of the digits in N and discards them, so that the remaining digits, in their original order, form a new integer. The sum of the digits in this new integer is 200.

If M is the largest integer that Stef could have formed, what are the first ten digits of M?





Problem of the Week

Problem E and Solution

Discarding Digits

Problem

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That is,

$N = 1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.$

Stef then selects some of the digits in N and discards them, so that the remaining digits, in their original order, form a new integer. The sum of the digits in this new integer is 200.

If M is the largest integer that Stef could have formed, what are the first ten digits of M?

Solution

We start by determining the sum of the digits of N. This is the same as determining the sum of the digits of the numbers from 1 to 50. The digits 0 to 9 sum to
$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

The digits in the numbers 10 to 19 sum to
$(1 + 0) + (1 + 1) + (1 + 2) + (1 + 3) + (1 + 4) + (1 + 5) + (1 + 6) + (1 + 7) + (1 + 8) + (1 + 9)$
$= 10(1) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 10 + 45 = 55$.

The digits in the numbers 20 to 29 sum to
$(2 + 0) + (2 + 1) + (2 + 2) + (2 + 3) + (2 + 4) + (2 + 5) + (2 + 6) + (2 + 7) + (2 + 8) + (2 + 9)$
$= 10(2) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 20 + 45 = 65$.

Similarly, the digits in the numbers 30 to 39 sum to $10(3) + 45 = 75$ and the digits in the numbers 40 to 49 sum to $10(4) + 45 = 85$.

We must add 5 and 0 in order to account for the number 50 at the end of N. Therefore, the sum of the digits of N is $45 + 55 + 65 + 75 + 85 + (5 + 0) = 330$.

Since the digits of M sum to 200, the digits that are removed and discarded must sum to $330 - 200 = 130$.

In order for M to be as large as possible, we need M to have as many digits as possible. So we need to remove as few digits as possible such that the digits that are removed sum to 130. To determine the fewest number of digits to remove, we remove the largest digits in N.

We notice that in N there are five 9s, five 8s, and five 7s. These 15 digits have a sum of $5 \times 9 + 5 \times 8 + 5 \times 7 = 120$. After removing these digits, we would still need to remove additional digits that have a sum of $130 - 120 = 10$. We would need at least two more digits to do this. So the fewest number of digits that we can remove that will have a sum of 130 is 17.

So which 17 digits do we remove? We are left with the following four options.

1. Remove five 9s, five 8s, five 7s, one 6, and one 4.
2. Remove five 9s, five 8s, five 7s, and two 5s.
3. Remove five 9s, five 8s, four 7s, two 6s, and one 5.
4. Remove five 9s, four 8s, five 7s, and three 6s.

These are the only ways to remove 17 digits from N that have a sum of 130. Thus, each option will result in a number that is exactly 17 digits shorter than N. So to determine which option results in the largest possible number, we can look at how each affects the first few digits of N.

Option 1: Remove five 9s, five 8s, five 7s, one 6, and one 4.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$.

- Removing a 6 and a 4 from anywhere after the first six digits will result in a number whose first six digits are 123456.
- Removing the first 6, and a 4 from anywhere past this 6 will result in a number whose first 6 digits are 123451.
- Removing the first 4, and a 6 from anywhere else in the number will result in a number whose first 6 digits are 123510 or 123561.

Since $123561 > 123510 > 123456 > 123451$, removing the first 4, and a 6 from anywhere else in the number can result in a number whose first six digits are 123561. This would result in the largest possible number so far.

Option 2: Remove five 9s, five 8s, five 7s, and two 5s.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$. No matter how we remove two 5s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 3: Remove five 9s, five 8s, four 7s, two 6s, and one 5.
After removing all the 9s and 8s, the remaining digits start $1234567101112\ldots$. No matter how we remove four 7s, two 6s, and one 5, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 4: Remove five 9s, four 8s, five 7s, and three 6s.
After removing all the 9s and 7s, the remaining digits start $1234568101112\ldots$. No matter how we remove four 8s, and three 6s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Therefore, in order to form the largest possible value of M, we should remove all 9s, 8s, and 7s, the first 4, and a 6 from anywhere else in the number.

After removing the 9s, 8s, 7s, and the first 4, we are left with

$$12356101112131415161112021222324252622230313233343536333404142434445464450$$

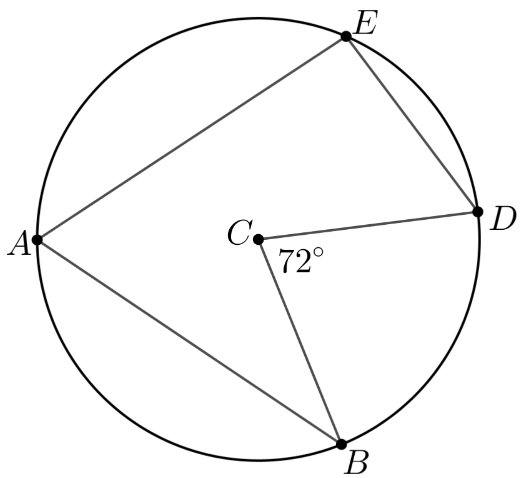
We still must remove a 6 to bring our digit sum to 200. We want whatever 6 we remove to affect the size of the final number in the least possible way. We need to therefore remove the 6 with the lowest place value. The 6 to be removed is therefore the hundred thousands digit in the number shown just above. After removing this 6,

$$M = 1235610111213141516111202122232425262223031323334353633340414243444544450$$

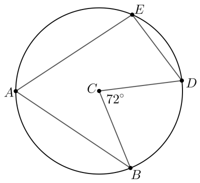
It follows that the first 10 digits of M are 1235610111.

Problem of the Week
Problem E
Find Another Angle

The points A, B, D, and E lie on the circumference of a circle with centre C, as shown.



If $\angle BCD = 72^\circ$ and $CD = DE$, then determine the measure of $\angle BAE$.



Problem of the Week

Problem E and Solution

Find Another Angle

Problem

The points A, B, D, and E lie on the circumference of a circle with centre C, as shown.

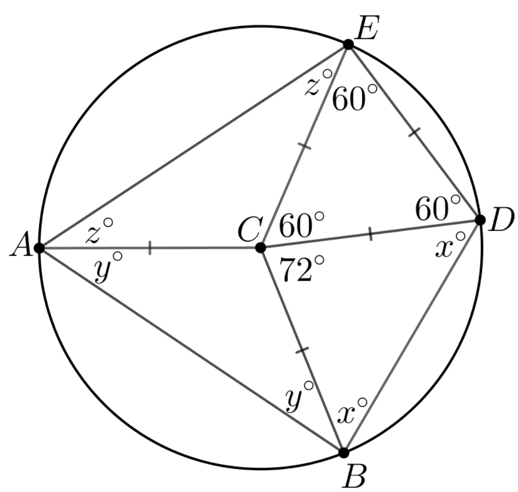
If $\angle BCD = 72^\circ$ and $CD = DE$, then determine the measure of $\angle BAE$.

Solution

We draw radii from C to points A and E on the circumference, and join B to D. Since CA, CB, CD, and CE are all radii, $CA = CB = CD = CE$.

We're given that $CD = DE$. Since $CD = CE$, we have $CD = CE = DE$, and thus $\triangle CDE$ is equilateral. It follows that $\angle ECD = \angle CED = \angle CDE = 60^\circ$.

Let $\angle CDB = x^\circ$, $\angle CBA = y^\circ$, and $\angle CAE = z^\circ$.
Since $CB = CD$, $\triangle CBD$ is isosceles. Therefore, $\angle CBD = \angle CDB = x^\circ$.
Since $CA = CB$, $\triangle CAB$ is isosceles. Therefore, $\angle CAB = \angle CBA = y^\circ$.
Since $CE = CA$, $\triangle CEA$ is isosceles. Therefore, $\angle CEA = \angle CAE = z^\circ$.



Since the angles in a triangle sum to 180°, from $\triangle CBD$ we have $x^\circ + x^\circ + 72^\circ = 180^\circ$. Thus, $2x^\circ = 108^\circ$ and $x = 54$.

Since $ABDE$ is a quadrilateral and the sum of the interior angles of a quadrilateral is equal to 360°, we have

$$\begin{aligned}
\angle BAE + \angle ABD + \angle BDE + \angle DEA &= 360^\circ \\
(y^\circ + z^\circ) + (y^\circ + x^\circ) + (x^\circ + 60^\circ) + (60^\circ + z^\circ) &= 360^\circ \\
2x + 2y + 2z &= 240 \\
x + y + z &= 120 \\
54 + y + z &= 120 \\
y + z &= 66
\end{aligned}$$

Since $\angle BAE = (y + z)^\circ$, then $\angle BAE = 66^\circ$.

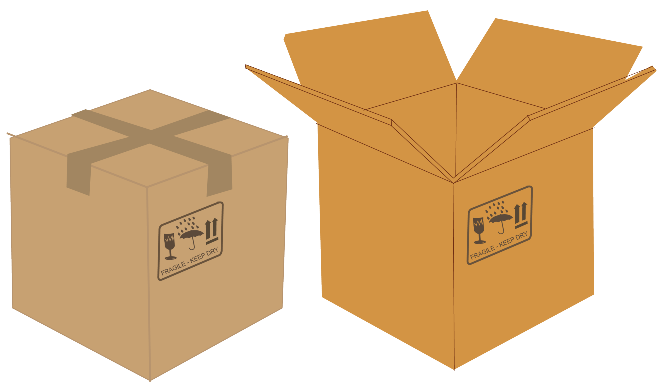
Problem of the Week
Problem E
Picking Boxes

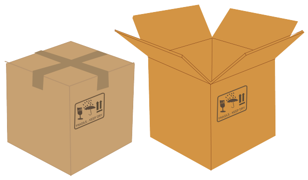
Billy's Box Company sells boxes with the following very particular restrictions on their dimensions.

- The length, width, and height, in cm, must be all integers.
- The ratio of the length to the width to the height must be 4 : 3 : 5.
- The sum of the length, width, and height must be between 100 cm and 1000 cm, inclusive.

Stefan bought the box with the smallest possible volume, and Lali bought the box with the largest volume less than 4 m^3.

Determine the dimensions of Stefan and Lali's boxes.





Problem of the Week
Problem E and Solution
Picking Boxes

Problem

Billy's Box Company sells boxes with the following very particular restrictions on their dimensions.

- The length, width, and height, in cm, must be all integers.
- The ratio of the length to the width to the height must be $4:3:5$.
- The sum of the length, width, and height must be between 100 cm and 1000 cm, inclusive.

Stefan bought the box with the smallest possible volume, and Lali bought the box with the largest volume less than 4 m^3.

Determine the dimensions of Stefan and Lali's boxes.

Solution

Since the boxes from Billy's Box Company have integer side lengths in the ratio $4:3:5$, let $4n$ represent the length of a box in cm, let $3n$ represent the width of a box in cm, and let $5n$ represent the height of a box in cm, where n is an integer.

Furthermore, the sum of the length, width and height must be at least 100 cm. It follows that

$$\begin{aligned} 4n + 3n + 5n &\geq 100 \\ 12n &\geq 100 \\ n &\geq \frac{100}{12} = 8\frac{1}{3} \end{aligned}$$

Also, the sum of the length, width and height must be at most 1000 cm. It follows that

$$\begin{aligned} 4n + 3n + 5n &\leq 1000 \\ 12n &\leq 1000 \\ n &\leq \frac{1000}{12} = 83\frac{1}{3} \end{aligned}$$

There is one other restriction to consider, since the volume of Lali's box is less than 4 m^3. To convert from m^3 to cm^3, note that

$$\begin{aligned} 1\text{ m}^3 &= 1\text{ m}\times 1\text{ m}\times 1\text{ m}\\ &= 100\text{ cm}\times 100\text{ cm}\times 100\text{ cm}\\ &= 1\,000\,000\text{ cm}^3 \end{aligned}$$

Therefore, $4\text{ m}^3 = 4\,000\,000\text{ cm}^3$.

It follows that

$$\begin{aligned} (4n)(3n)(5n) &< 4\,000\,000\\ 60n^3 &< 4\,000\,000\\ n^3 &< \frac{200\,000}{3}\\ n &< \sqrt[3]{\frac{200\,000}{3}} \approx 40.5 \end{aligned}$$

We also know that n is an integer. Since $n \geq 8\frac{1}{3}$, then the smallest possible integer value of n is 9. Using the dimensions $4n$, $3n$, and $5n$ with $n = 9$, we can determine that the dimensions of Stefan's box are 36 cm by 27 cm by 45 cm.

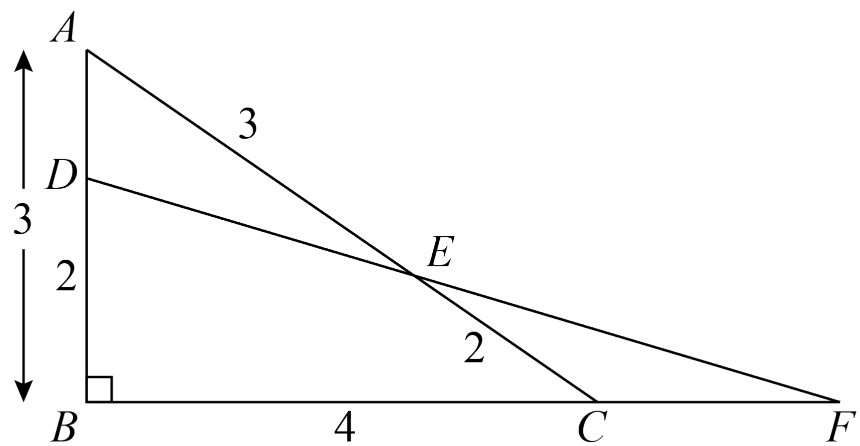
For Lali's box, since $n \leq 83\frac{1}{3}$ and $n < 40.5$, then the largest possible value of n is 40. Using the dimensions $4n$, $3n$, and $5n$ with $n = 40$, we can determine that the dimensions of Lali's box are 160 cm by 120 cm by 200 cm. This box has a volume of 3.84 m^3.

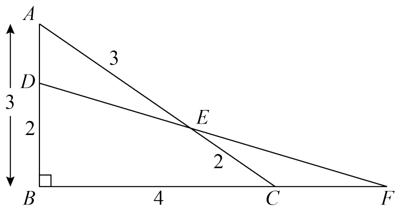
Problem of the Week

Problem E

Overlapping Shapes 3

Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90^\circ$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that D lies on AB, F lies on the extension of BC, $DB = 2$ cm, and sides AC and DF meet at E. If $AE = 3$ cm and $EC = 2$ cm, determine the length of CF.





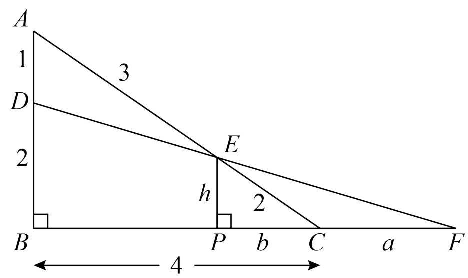
Problem of the Week
Problem E and Solution
Overlapping Shapes 3

Problem

Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90°$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that D lies on AB, F lies on the extension of BC, $DB = 2$ cm, and sides AC and DF meet at E. If $AE = 3$ cm and $EC = 2$ cm, determine the length of CF.

Solution

Since $AB = 3$ and $DB = 2$, it follows that $AD = 1$ cm. Draw a perpendicular from E to BF.
Let P be the point where the perpendicular intersects BF.
Let $CF = a$, $PC = b$, and $EP = h$.
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles, the second uses trigonometry, and the third uses coordinate geometry.



Solution 1

Since EP is perpendicular to BF, we know $\angle EPF = 90°$. Also, $\angle ECP = \angle ACB$ (same angle). Therefore, $\triangle ABC \sim \triangle EPC$ (by angle-angle triangle similarity).

From the similarity, $\dfrac{AC}{BC} = \dfrac{EC}{PC}$, so $\dfrac{5}{4} = \dfrac{2}{b}$ or $b = \dfrac{8}{5}$. Also, $\dfrac{AC}{AB} = \dfrac{EC}{EP}$, so $\dfrac{5}{3} = \dfrac{2}{h}$ or $h = \dfrac{6}{5}$.

Now let's calculate PF. We know $\angle EPF = \angle DBF = 90°$ and $\angle EFP = \angle DFB$ (same angle).

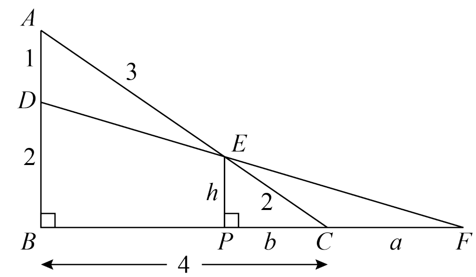
Therefore, $\triangle DBF \sim \triangle EPF$ (by angle-angle triangle similarity). This tells us $\dfrac{DB}{BF} = \dfrac{EP}{PF}$.

Since $BF = BC + CF = 4 + a$ and $PF = PC + CF = \frac{8}{5} + a$, we have

$$\begin{aligned}
\frac{DB}{BF} &= \frac{EP}{PF} \\
\frac{2}{4+a} &= \frac{\frac{6}{5}}{\frac{8}{5}+a} \\
\frac{16}{5} + 2a &= \frac{24}{5} + \frac{6}{5}a \\
2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5} \\
\frac{4}{5}a &= \frac{8}{5} \\
a &= 2
\end{aligned}$$

Therefore, $CF = 2$ cm.

Solution 2



In $\triangle EPC$, $\sin(\angle ECP) = \frac{h}{2}$. In $\triangle ABC$, $\sin(\angle ACB) = \frac{3}{5}$.
Since $\angle ECP = \angle ACB$ (same angle),

$$\sin(\angle ECP) = \sin(\angle ACB)$$
$$\frac{h}{2} = \frac{3}{5}$$
$$h = \frac{6}{5}$$

Since $\triangle EPC$ is a right-angled triangle,

$$EP^2 + PC^2 = EC^2$$
$$h^2 + b^2 = 2^2$$
$$\left(\frac{6}{5}\right)^2 + b^2 = 4$$
$$b^2 = 4 - \frac{36}{25}$$
$$b^2 = \frac{64}{25}$$
$$b = \frac{8}{5}, \quad \text{since} \quad b > 0$$

In $\triangle EPF$, $\tan(\angle EFP) = \dfrac{EP}{PF} = \dfrac{h}{a+b} = \dfrac{\frac{6}{5}}{a+\frac{8}{5}}$.

In $\triangle DBF$, $\tan(\angle DFB) = \dfrac{DB}{BF} = \dfrac{2}{4+a}$.

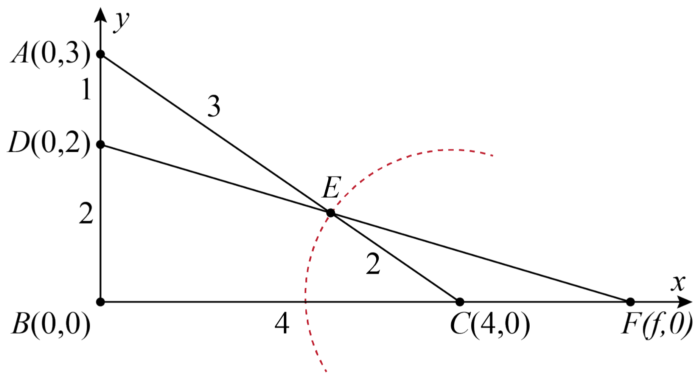
Since $\angle EFP = \angle DFB$ (same angle),

$$\tan(\angle EFP) = \tan(\angle DFB)$$
$$\frac{\frac{6}{5}}{a+\frac{8}{5}} = \frac{2}{4+a}$$
$$\frac{24}{5} + \frac{6}{5}a = 2a + \frac{16}{5}$$
$$2a - \frac{6}{5}a = \frac{24}{5} - \frac{16}{5}$$
$$\frac{4}{5}a = \frac{8}{5}$$
$$a = 2$$

Therefore, $CF = 2$ cm.

Solution 3

We will use coordinate geometry in this solution, and place B at the origin. Using the given information, D is at $(0,2)$, A is at $(0,3)$, C is at $(4,0)$, and F is on the positive x-axis at $(f,0)$ with $f>4$. Consider the circle through E with centre $C(4,0)$. Since $CE=2$, the radius of this circle is 2. Thus, the equation of this circle is $(x-4)^2+y^2=4$.



The line passing through $A(0,3)$ and $C(4,0)$ has y-intercept 3 and slope $-\frac{3}{4}$, and so has equation $y=-\frac{3}{4}x+3$. Since E lies on the line with equation $y=-\frac{3}{4}x+3$ and the circle with equation $(x-4)^2+y^2=4$, to find the coordinates of E, we substitute $y=-\frac{3}{4}x+3$ for y in $(x-4)^2+y^2=4$. Note that E is in the first quadrant so $x>0$ and $y>0$.

Doing so, we get

$$(x-4)^2+\left(-\frac{3}{4}x+3\right)^2=4$$

Expanding the left side, we get

$$x^2-8x+16+\frac{9}{16}x^2-\frac{9}{2}x+9=4$$

Multiplying by 16, we get

$$16x^2-128x+256+9x^2-72x+144=64$$

Simplifying, we get

$$25x^2-200x+336=0$$

Factoring, we then get

$$(5x-12)(5x-28)=0$$

It follows that $x=\frac{12}{5}$ or $x=\frac{28}{5}$. Substituting $x=\frac{12}{5}$ in $y=-\frac{3}{4}x+3$, we obtain $y=\frac{6}{5}$.

Substituting $x=\frac{28}{5}$ in $y=-\frac{3}{4}x+3$, we obtain $y=-\frac{6}{5}$. But E is in the first quadrant so $y>0$, and this second possibility is inadmissible. It follows that E has coordinates $\left(\frac{12}{5},\frac{6}{5}\right)$.

We can now find the equation of the line containing $D(0,2)$, $E\left(\frac{12}{5},\frac{6}{5}\right)$, and $F(f,0)$. This line has y-intercept 2, slope equal to $\dfrac{\frac{6}{5}-2}{\frac{12}{5}-0}=\dfrac{-\frac{4}{5}}{\frac{12}{5}}=-\frac{1}{3}$, and thus has equation $y=-\frac{1}{3}x+2$.

The point $F(f,0)$ lies on this line, so $0=-\frac{1}{3}(f)+2$, which leads to $f=6$. Thus, the point F has coordinates $(6,0)$. Since C is at $(4,0)$ and F is at $(6,0)$, $CF=2$. It turns out that F also lies on the circle through E.

Therefore, $CF=2$ cm.

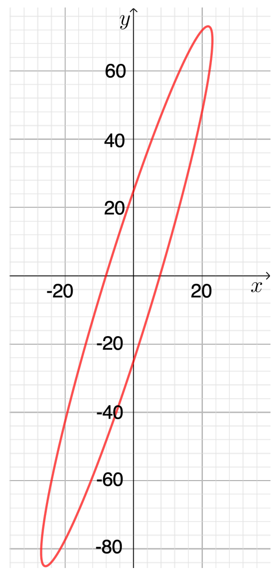
Problem of the Week

Problem E

Points on an Ellipse

The graph of $(x+1)^2+(y-2)^2=100$ is a circle with centre $(-1,2)$ and radius 10.

The graph of $10x^2-6xy+4x+y^2=621$ is shown below. The shape of this curve is known as an ellipse.



List all the ordered pairs (x,y) of non-negative integers x and y that satisfy the equation $10x^2-6xy+4x+y^2=621$.

Note: When solving this problem, it might be useful to use the following idea.

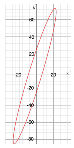
By completing the square,

$$x^2+y^2+2x-4y=95$$

can be rewritten as

$$(x+1)^2+(y-2)^2=100$$

One solution to this equation is $(x,y)=(5,10)$.



Problem of the Week

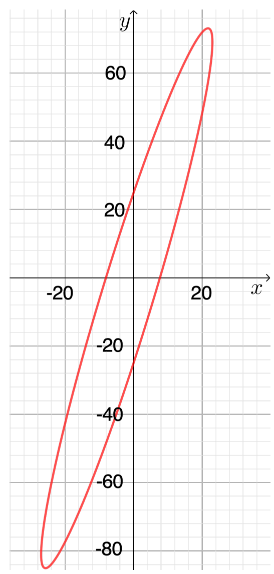
Problem E and Solution

Points on an Ellipse

Problem

The graph of $(x+1)^2 + (y-2)^2 = 100$ is a circle with centre $(-1, 2)$ and radius 10.

The graph of $10x^2 - 6xy + 4x + y^2 = 621$ is shown below. The shape of this curve is known as an ellipse.



List all the ordered pairs (x, y) of non-negative integers x and y that satisfy the equation $10x^2 - 6xy + 4x + y^2 = 621$.

Note: When solving this problem, it might be useful to use the following idea.

By completing the square,

$$x^2 + y^2 + 2x - 4y = 95$$

can be rewritten as

$$(x+1)^2 + (y-2)^2 = 100$$

One solution to this equation is $(x, y) = (5, 10)$.

Solution

Starting with the given equation, we obtain the following equivalent equations:

$$\begin{aligned} 10x^2 - 6xy + 4x + y^2 &= 621 \\ 9x^2 - 6xy + y^2 + x^2 + 4x &= 621 \\ 9x^2 - 6xy + y^2 + x^2 + 4x + 4 &= 621 + 4 \\ (3x - y)^2 + (x + 2)^2 &= 625 \end{aligned}$$

Notice that $625 = 25^2$.
Since x and y are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625 = 25^2$.

The pairs of perfect squares that sum to 625 are 625 and 0, 576 and 49, and 400 and 225.

Therefore, $(3x - y)^2$ and $(x + 2)^2$ are equal to 25^2 and 0^2 in some order, or 24^2 and 7^2 in some order, or 20^2 and 15^2 in some order.

Furthermore, $3x - y$ and $x + 2$ are equal to ± 25 and ± 0 in some order, or ± 24 and ± 7 in some order, or ± 20 and ± 15 in some order.

Since $x \geq 0$, then $x + 2 \geq 2$. So we need to consider when $x + 2$ is equal to 25, 24, 7, 20, or 15.

- If $x + 2 = 25$, then $x = 23$. Also, $3x - y = 0$. Thus, $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(23, 69)$ is a valid ordered pair.
- If $x + 2 = 24$, then $x = 22$. Also, $3x - y = 7$ or $3x - y = -7$.
When $3x - y = 7$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(22, 59)$ is a valid ordered pair.
When $3x - y = -7$, we find $y = 73$. Since $x \geq 0$ and $y \geq 0$, $(22, 73)$ is a valid ordered pair.
- If $x + 2 = 7$, then $x = 5$. Also, $3x - y = 24$ or $3x - y = -24$.
When $3x - y = 24$, we find $y = -9$. Since $y < 0$, this does not lead to a valid ordered pair.
When $3x - y = -24$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(5, 39)$ is a valid ordered pair.
- If $x + 2 = 20$, then $x = 18$. Also, $3x - y = 15$ or $3x - y = -15$.
When $3x - y = 15$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(18, 39)$ is a valid ordered pair.
When $3x - y = -15$, we find $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(18, 69)$ is a valid ordered pair.
- If $x + 2 = 15$, then $x = 13$. Also, $3x - y = 20$ or $3x - y = -20$.
When $3x - y = 20$, we find $y = 19$. Since $x \geq 0$ and $y \geq 0$, $(13, 19)$ is a valid ordered pair.
When $3x - y = -20$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(13, 59)$ is a valid ordered pair.

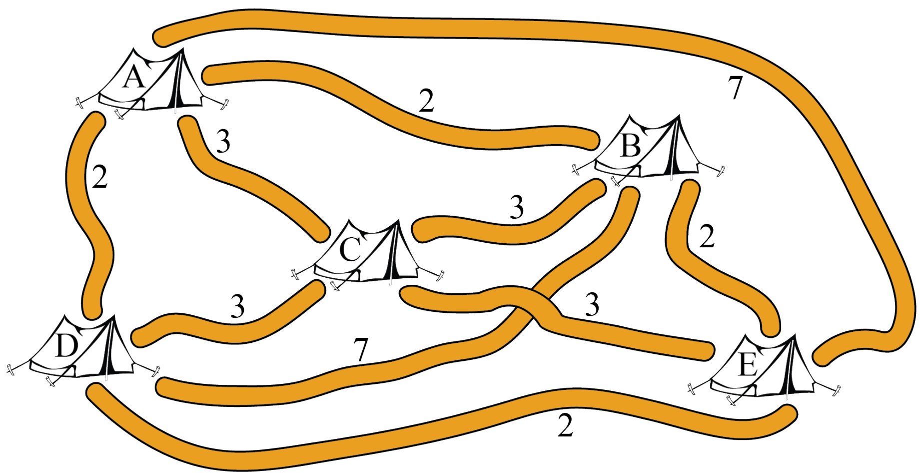
Therefore, the ordered pairs of non-negative integers that satisfy the equation are $(23, 69)$, $(22, 59)$, $(22, 73)$, $(5, 39)$, $(18, 39)$, $(18, 69)$, $(13, 19)$, and $(13, 59)$.

Computational Thinking (C)

Take me to the cover

Problem of the Week
Problem E
In Tents

Five tents, each of a different colour, are to be placed on five different campsites. The five campsites are arranged as shown, with each pair of campsites connected by a path. The number on each path indicates the number of minutes it takes to walk along that path.



Jared wants to start at the blue tent, and take a walk along some of the paths passing the tents in the order green, white, yellow, red, and yellow, before returning to the blue tent.

If Jared wants the total time that he spends walking to be as small as possible, what colour tent(s) should be put in campsite C?

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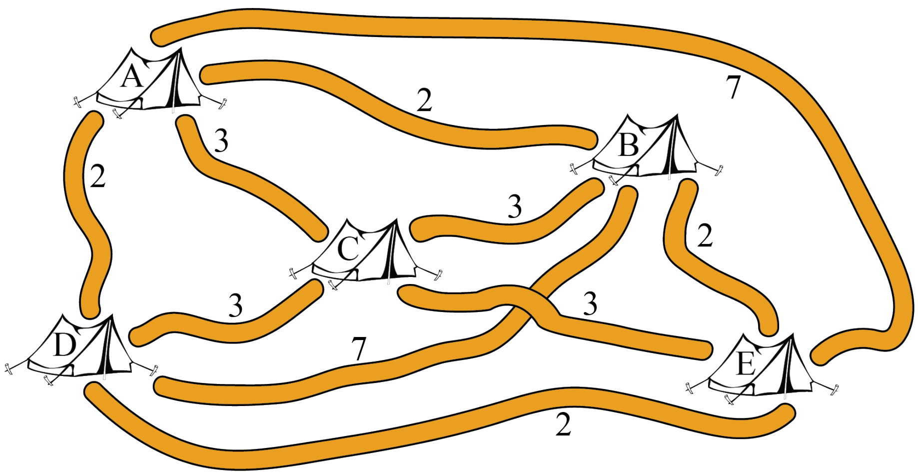
This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem E and Solution
In Tents

Problem

Five tents, each of a different colour, are to be placed on five different campsites. The five campsites are arranged as shown, with each pair of campsites connected by a path. The number on each path indicates the number of minutes it takes to walk along that path.



Jared wants to start at the blue tent, and take a walk along some of the paths passing the tents in the order green, white, yellow, red, and yellow, before returning to the blue tent.

If Jared wants the total time that he spends walking to be as small as possible, what colour tent(s) should be put in campsite C?

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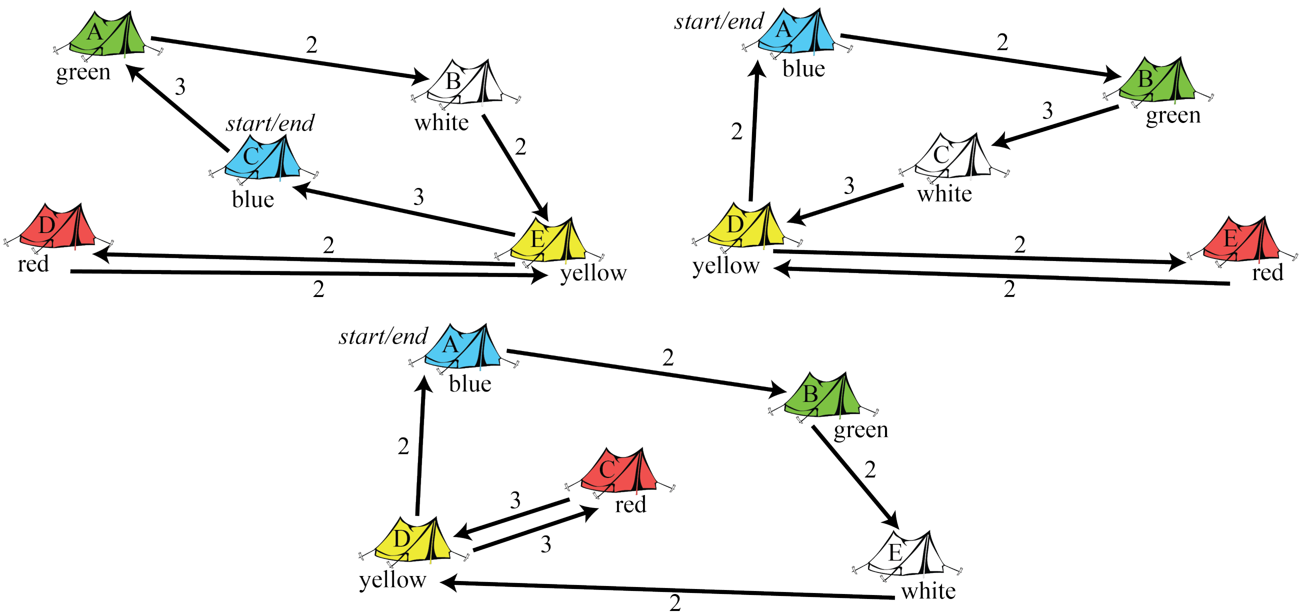
This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Solution

The given sequence of tent colours is blue, green, white, yellow, red, yellow, and blue. This tells us that Jared will pass by all five tents and walk along the following six paths: blue $\rightarrow$ green, green $\rightarrow$ white, white $\rightarrow$ yellow, yellow $\rightarrow$ red, red $\rightarrow$ yellow, and yellow $\rightarrow$ blue. In this sequence, Jared will walk to and from each campsite at least once. Therefore, he needs to walk to and from campsite C at least once. So the minimum possible time occurs when Jared takes two of the 3 min paths that connect to campsite C, and four of the 2 min paths. Such a walk would take Jared $2(3) + 4(2) = 14$ min.

Three such routes are shown, where the tent in campsite C is blue, white, and red, respectively.



Now we will show why it is not possible for the tent in campsite C to be yellow or green if Jared's walk takes 14 min in total.

If the tent in campsite C is yellow, then Jared will have to walk on four 3 min paths because he passes the yellow tent twice on his walk. Then the minimum possible time would be $4(3) + 2(2) = 16$ min. This is more than 14 min. Thus, if the tent in campsite C is yellow, then Jared's walk can't take the minimum time of 14 min.

If the tent in campsite C is green, then in order to achieve the minimum time of 14 min, the paths white $\rightarrow$ yellow, yellow $\rightarrow$ red, and yellow $\rightarrow$ blue must each take 2 min. However this is not possible because each of the remaining campsites has 2 min paths connecting to only two other campsites. So the yellow tent cannot have a 2 min path connecting to each of the white, red, and blue tents. This means that one of these paths must take 7 min, which would mean the minimum possible time would be $2(3) + 3(2) + 7 = 19$ min. This is more than 14 min. Thus, if the tent in campsite C is green, then Jared's walk can't take the minimum time of 14 min.

Therefore, the tent in campsite C should be blue, white, or red.

Problem of the Week
Problem E
Coin Combinations

In Canada, a \$2 coin is called a toonie, a \$1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of \$1.

How many different combinations of toonies, loonies, and/or quarters have a total value of \$100?

NOTE: In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For example, the sum of the first 10 positive integers is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{10(10+1)}{2} = \frac{10(11)}{2} = 55$$

Problem of the Week
Problem E and Solution
Coin Combinations

Problem

In Canada, a \$2 coin is called a toonie, a \$1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of \$1.

How many different combinations of toonies, loonies, and/or quarters have a total value of \$100?

Solution

We will break the solution into cases based on the number of \$2 coins used. For each case, we will count the number of possibilities for the number of \$1 and 25¢ coins.

The maximum number of \$2 coins we can use is 50, since $\$2 \times 50 = \$100$. If we use 50 \$2 coins, then we do not need any \$1 or 25¢ coins. Therefore, there is only one way to make a total of \$100 if there are 50 \$2 coins.

Suppose we use 49 \$2 coins. Since $\$2 \times 49 = \$98$, to reach a total of \$100, we would need two \$1 and no 25¢ coins, or one \$1 and four 25¢ coins, or no \$1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of \$100 if we use 49 \$2 coins.

Suppose we use 48 \$2 coins. Since $\$2 \times 48 = \$96$, to reach a total of \$100, we would need four \$1 and no 25¢ coins, or three \$1 and four 25¢ coins, or two \$1 and eight 25¢ coins, or one \$1 and twelve 25¢ coins, or no \$1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of \$100 if we use 48 \$2 coins.

We start to see a pattern. When we reduce the number of \$2 coins by one, the number of possible combinations using that many \$2 coins increases by 2. This is because there are 2 more options for the number of \$1 coins we can use. Thus, when we use 47 \$2 coins, there are 7 possible ways to make a total of \$100. When we use 46 \$2 coins, there are 9 possible ways to make a total of \$100, and so on. When we use 1 \$2 coin, there are 99 different ways to make the difference of \$98 (because you can use 0 to 98 \$1 coins). When we don't use any \$2 coins, there are 101 different ways to make a total of \$100 (because you can use 0 to 100 \$1 coins). Thus, the number of different combinations of coins that have a total value of \$100 is

$$1 + 3 + 5 + 7 + 9 + \cdots + 99 + 101$$

Adding and subtracting the even numbers from 2 to 100, we get

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \cdots + 98 + 99 + 100 + 101 - (2 + 4 + 6 + 8 + \cdots + 98 + 100)$$

Factoring out a factor of 2 from the subtracted even numbers, we get

$$(1 + 2 + 3 + \cdots + 100 + 101) - 2(1 + 2 + 3 + 4 + \cdots + 50)$$

We can then use the formula for the sum of the first n positive integers to find that this expression is equal to

$$\frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) = 101(51) - 50(51) = 2601$$

Therefore, there are 2601 different combinations of toonies, loonies, and/or quarters that have a total value of \$100.

Extension:

Let's look at the end of the previous computation another way.

$$\begin{aligned} 1+3+5+7+9+\cdots+99+101 &= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) \\ &= 101(51) - 50(51) \\ &= 51(101-50) \\ &= 51(51) \\ &= 51^2 \end{aligned}$$

How many odd integers are in the list from 1 to 101? From 1 to 101, there are 101 integers. This list contains the even integers, from 2 to 100, which are 50 in total. Therefore, there are $101 - 50 = 51$ odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is equal to 51^2? Is the sum of the first 1000 odd positive integers equal to 1000^2? Is the sum of the first n odd positive integers equal to n^2?

We will develop a formula for the sum of the first n odd positive integers.

We saw in the problem statement that the sum of the first n positive integers is

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Every odd positive integer can be written in the form $2n-1$, where n is an integer ≥ 1. When $n = 1,\ 2n-1 = 2(1)-1 = 1$; when $n = 2,\ 2n-1 = 2(2)-1 = 3$, and so on. So the 51^{st} odd positive integer is $2(51)-1 = 101$, as we determined above. The n^{th} odd positive integer is $2n-1$. Let's consider the sum of the first n odd positive integers. That is,

$$1+3+5+7+\cdots+(2n-3)+(2n-1)$$

Adding and subtracting the even numbers from 2 to $2n$, we get

$$\begin{aligned} &1+2+3+4+5+\cdots+(2n-3)+(2n-2)+(2n-1)+2n-(2+4+6+\cdots+(2n-2)+2n) \\ &= (1+2+3+4+\cdots+2n)-(2+4+6+8+\cdots+(2n-2)+2n) \end{aligned}$$

Factoring out a 2 from the subtracted even numbers, we get

$$(1+2+3+4+\cdots+2n)-2(1+2+3+\cdots+n)$$

We can then use the formula for the sum of the first n positive integers to find that this expression is equal to

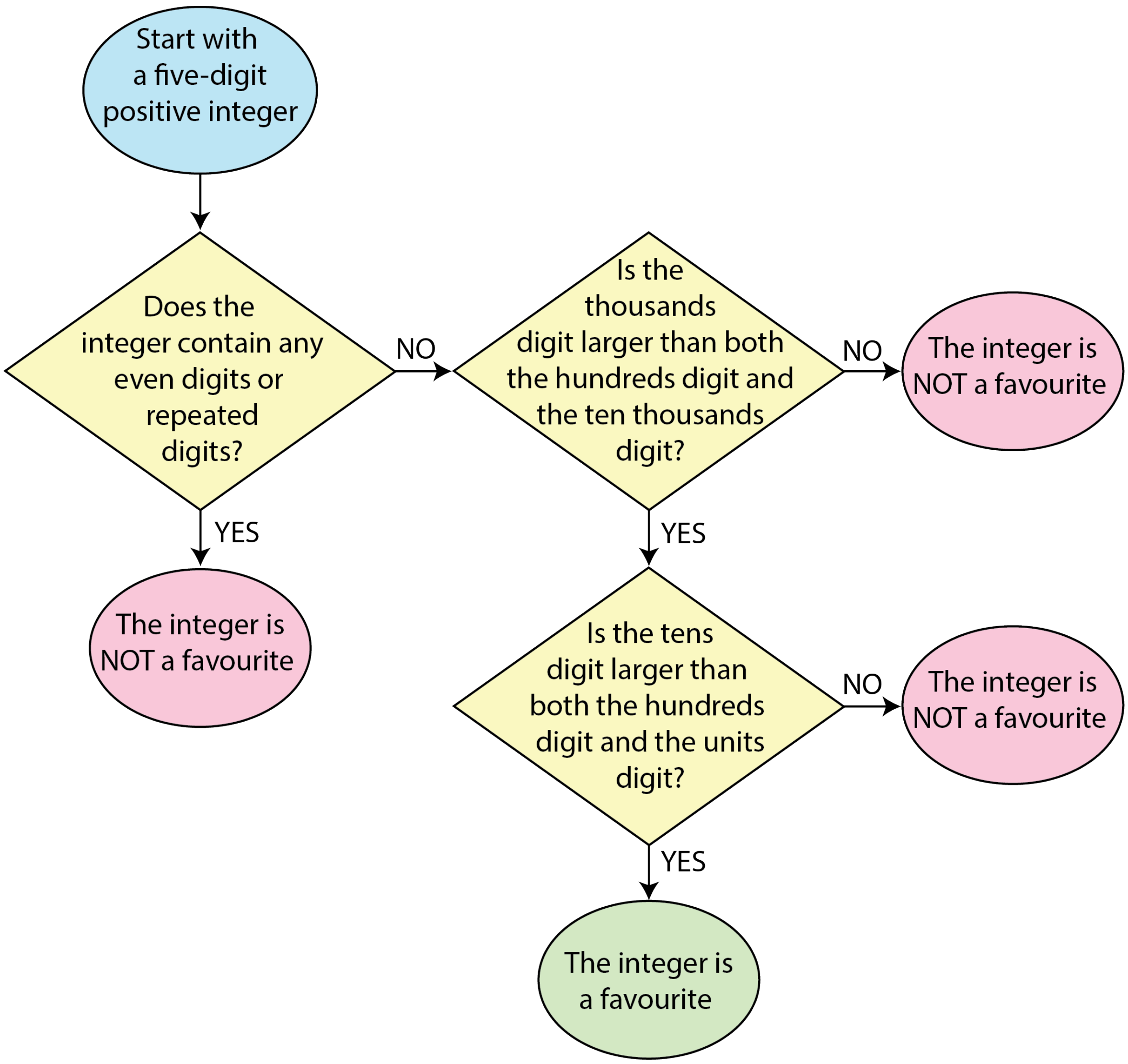
$$\begin{aligned} \frac{2n(2n+1)}{2} - 2\left(\frac{n(n+1)}{2}\right) &= n(2n+1)-n(n+1) \\ &= 2n^2+n-n^2-n \\ &= n^2 \end{aligned}$$

Therefore, the sum of the first n odd positive integers is equal to n^2.

For Further Thought: Can you develop a formula for the sum of the first n even positive integers?

Problem of the Week
Problem E
Favourite Numbers

Adrian likes all the numbers, but some are his favourites. He created a flowchart to help people determine whether or not a given five-digit positive integer is one of his favourites.

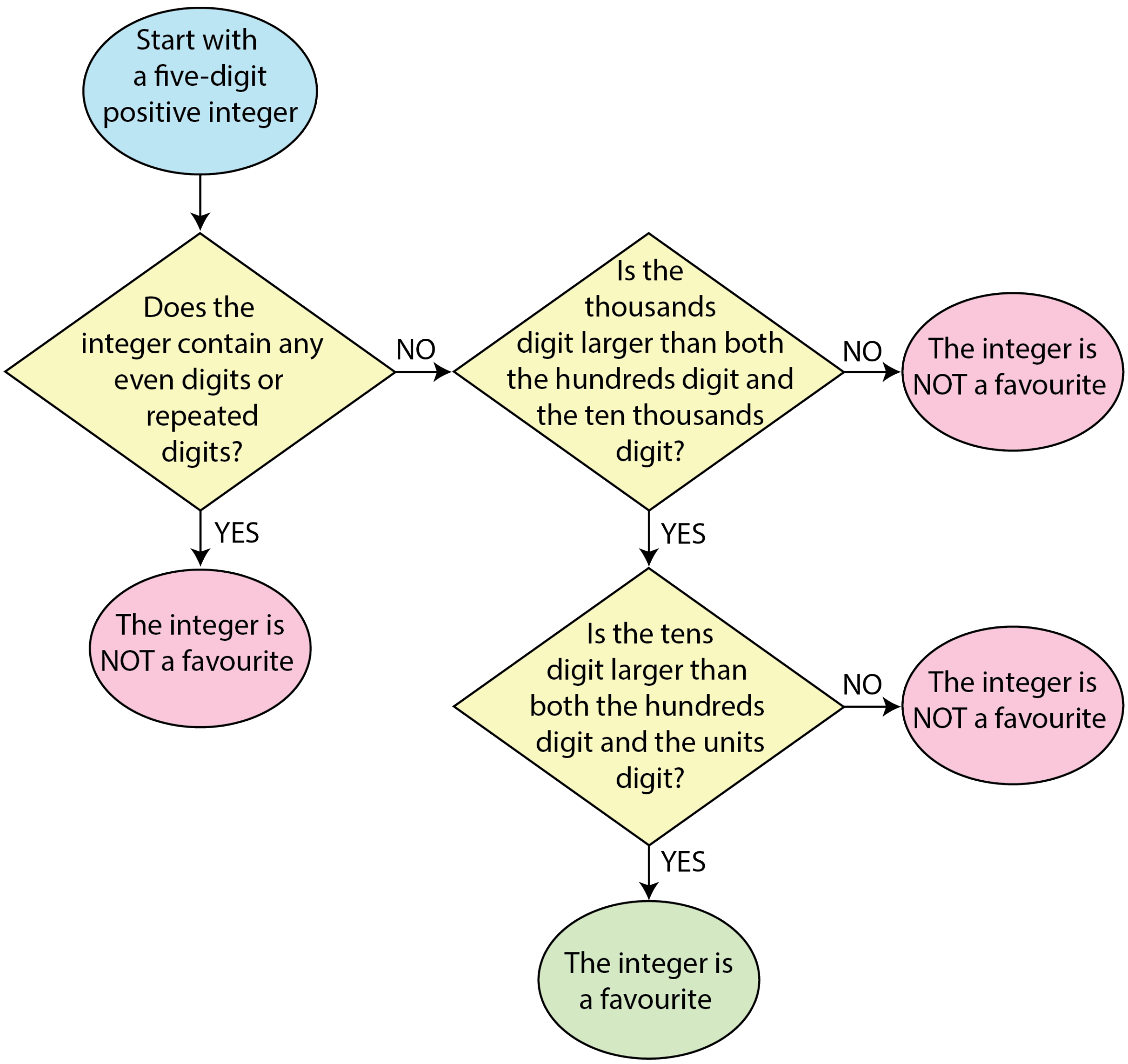


How many favourite five-digit positive integers does Adrian have?

Problem of the Week
Problem E and Solution
Favourite Numbers

Problem

Adrian likes all the numbers, but some are his favourites. He created a flowchart to help people determine whether or not a given five-digit positive integer is one of his favourites.



How many favourite five-digit positive integers does Adrian have?

Solution

We write one of Adrian's favourite five-digit positive integers as $VWXYZ$, where each letter represents a digit.

Since this integer does not contain any even or repeated digits, then it is created using the digits $1,\ 3,\ 5,\ 7$, and 9, in some order. We want to count the number of ways of assigning $1,\ 3,\ 5,\ 7,\ 9$ to the digits $V,\ W,\ X,\ Y,\ Z$ so that the answers to the second two questions in the flowchart are both yes.

Since the thousands digit is larger than both the hundreds digit and the ten thousands digit, then $W > X$ and $W > V$. Since the tens digit is larger than both the hundreds digit and the units digit, then $Y > X$ and $Y > Z$.

The digits 1 and 3 cannot be placed as W or Y, since W and Y are larger than both of their neighbouring digits, while 1 is smaller than all of the other digits and 3 is larger than only one of the other possible digits.

The digit 9 cannot be placed as $V,\ X$, or Z since it is the largest possible digit and so cannot be smaller than W or Y . Thus, 9 must be placed as W or as Y. Therefore, the digits W and Y are 9 and either 5 or 7.

Suppose that $W = 9$ and $Y = 5$. The number is thus $V9X5Z$. Neither X or Z can equal 7 since $7 > 5$, so $V = 7$. It follows that X and Z are 1 and 3, or 3 and 1. There are 2 possible integers in this case. Similarly, if $Y = 9$ and $W = 5$, there are 2 possible integers.

Suppose that $W = 9$ and $Y = 7$. The number is thus $V9X7Z$. The digits $1,\ 3$, and 5 can then be placed in any of the remaining spots. There are 3 choices for the digit V. For each of these choices, there are 2 choices for X, and then 1 choice for Z. There are thus $3 \times 2 \times 1 = 6$ possible integers in this case. Similarly, if $Y = 9$ and $W = 7$, there are 6 possible integers.

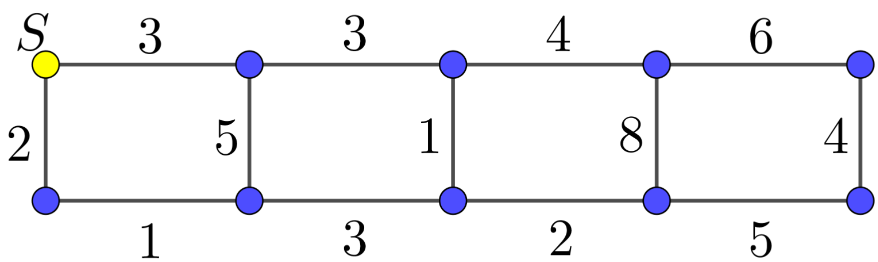
Therefore, Adrian has $2 + 2 + 6 + 6 = 16$ favourite positive five-digit integers.

Problem of the Week

Problem E

Special Delivery

In Gridville, the POTW Delivery Company needs to deliver nine packages to nine different locations. In the diagram below, the location where they start is labelled with an S. The nine other circles indicate where the nine delivery locations are. These locations are joined by roads, which are shown as lines. The number beside each line indicates the average time, in minutes, it will take to travel along that road.



If the POTW Delivery Company does not want to visit any location more than once, but can finish at any location, what is the shortest amount of time that they will take to deliver the nine packages?

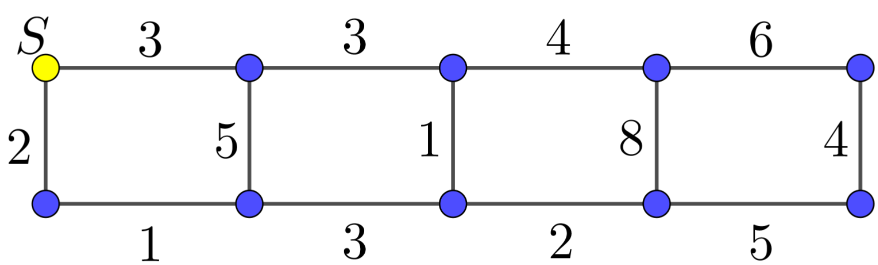
Problem of the Week

Problem E and Solution

Special Delivery

Problem

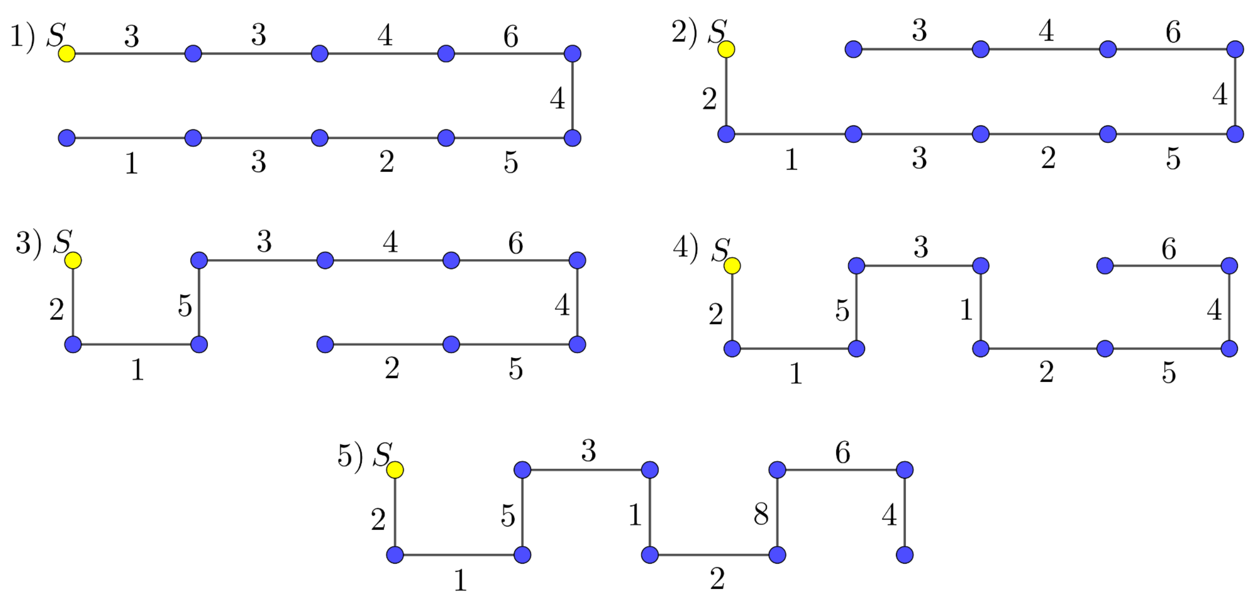
In Gridville, the POTW Delivery Company needs to deliver nine packages to nine different locations. In the diagram below, the location where they start is labelled with an S. The nine other circles indicate where the nine delivery locations are. These locations are joined by roads, which are shown as lines. The number beside each line indicates the average time, in minutes, it will take to travel along that road.



If the POTW Delivery Company does not want to visit any location more than once, but can finish at any location, what is the shortest amount of time that they will take to deliver the nine packages?

Solution

There are only five possible paths that start at S and visit each of the locations exactly once:



To justify this, notice that we have two choices for where to travel from S: down or right.

- If we travel right first, then the rest of our path is determined. If we move down before reaching the rightmost location in the top row, then we would need to backtrack to reach all of the bottom locations. This would mean revisiting at least one location on the way. Therefore, we must travel right until we reach the rightmost location in the top row, and then must follow path 1) for the remainder.
- If we travel down first, then we must travel right next, but then there we again have two options: up or right.
  - If we travel right, then the rest of our path is determined. We cannot move up next, as then we cannot reach all of the top locations without backtracking, and so we must move right. From here we would have to follow path 2) for the remainder.
  - If we travel up, then we must travel right next. From here we once again we have two choices: down or right.
    - If we travel right, then the rest of our path is determined. We must follow path 3).
    - If we travel down, then we must travel right next, and then we once more have two choices: up or right.
      - If we travel right then the rest of our path is determined. We must follow path 4).
      - If we travel up then the rest of our path is determined. We must follow path 5).

These five possible paths have the total times of:

1) $3 + 3 + 4 + 6 + 4 + 5 + 2 + 3 + 1 = 31$
2) $2 + 1 + 3 + 2 + 5 + 4 + 6 + 4 + 3 = 30$
3) $2 + 1 + 5 + 3 + 4 + 6 + 4 + 5 + 2 = 32$
4) $2 + 1 + 5 + 3 + 1 + 2 + 5 + 4 + 6 = 29$
5) $2 + 1 + 5 + 3 + 1 + 2 + 8 + 6 + 4 = 32$

(where the sum is shown starting at S and moving through the path).

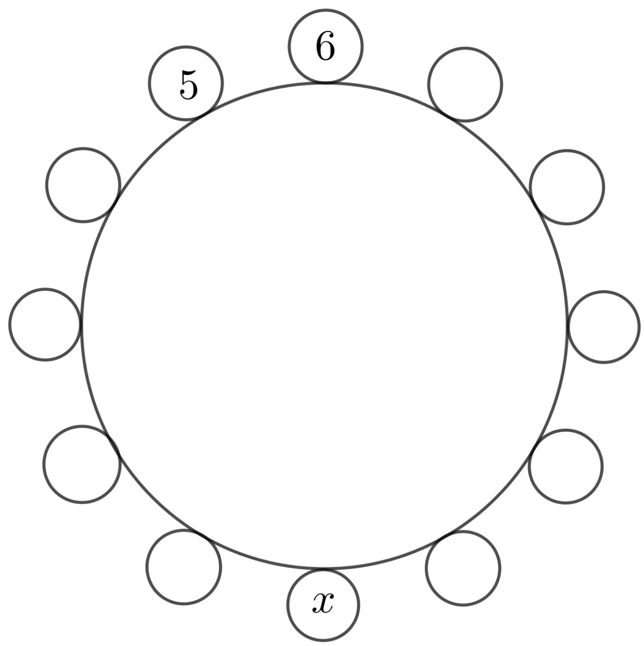
Therefore, the shortest amount of time to deliver the nine packages is 29 minutes.

Problem of the Week

Problem E

Take a Seat 3

Twelve people are sitting, equally spaced, around a circular table. They each hold a card with a different integer on it. For any two people sitting beside each other, the positive difference between the integers on their cards is no more than 2. The people holding the integers 5 and 6 are seated as shown. The person opposite the person holding the 6 is holding the integer x. What are the possible values of x?



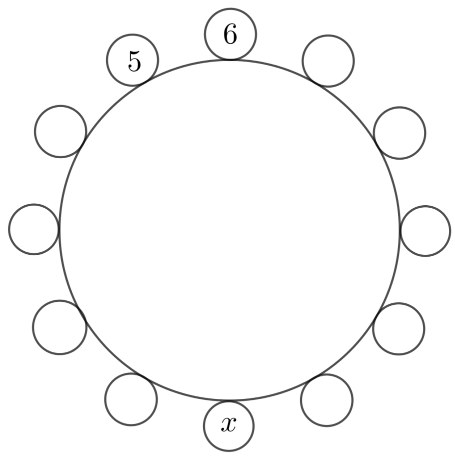
Problem of the Week

Problem E and Solution

Take a Seat 3

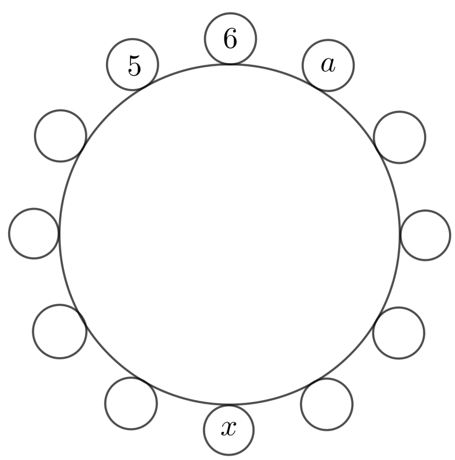
Problem

Twelve people are sitting, equally spaced, around a circular table. They each hold a card with a different integer on it. For any two people sitting beside each other, the positive difference between the integers on their cards is no more than 2. The people holding the integers 5 and 6 are seated as shown. The person opposite the person holding the 6 is holding the integer x. What are the possible values of x?



Solution

We will start with the card with the integer 6. We are given that 5 is on one side of the 6. Let a be the integer on the other side of the 6.



Since each card contains a different integer and the positive difference between the integers on two cards beside each other is no more than 2, then a must be 4, 7, or 8. We will consider these three cases.

Case 1: $a = 7$

Since the number on each card is different and we know that someone is holding a card with a 5 and someone is holding a card with a 6, then the integer to the right of the 7 must be 8 or 9.

Furthermore, every integer to the right of 7 must be greater than 7. Similarly, the integer to left of 5 is either 3 or 4. Furthermore, any integer to the left of 5 must be less than 5.

Since x is both to the right of 7 and to the left of 5, it must be both greater than 7 and less than 5. This is not possible.

Therefore, when $a = 7$, there is no solution for x.

Case 2: $a = 4$

Since the number on each card is different and we know that someone is holding a card with a 5 and someone is holding a card with a 6, then the integer to the right of 4 must be 3 or 2. We will look at these two subcases.

- **Case 2a:** The card with integer 3 is to the right of the card with integer 4.

  Notice then that every integer to the right of the 3 must be less than 3. Also the integer to the left of 5 must be 7 and every integer to left of the 7 must be greater than 7. Since x is both to the right of 3 and to the left of 7, it must be both greater than 7 and less than 3. This is not possible.

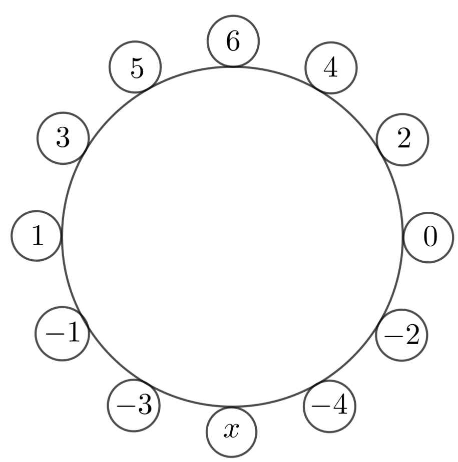
  Therefore, when $a = 4$ and the integer to the right of it is 3, there is no solution for x.

- **Case 2b:** The card with integer 2 is to the right of the card with integer 4.

  Now, the integer to the left of 5 can be either 7 or 3.

  If the integer is 7, then using a similar argument to that in Case 2a, there is no solution for x.

  If the integer to the left of 5 is 3, the only possible integer to the left of 3 is 1. This means the only possible integer to the right of 2 is 0. Which leads to the only possible integer to the left of 1 is -1. Furthermore, the only possible integer to the right of 0 is -2. Continuing in this manner, we get the table set up shown below.



  From here, the only possible solution is $x = -5$.

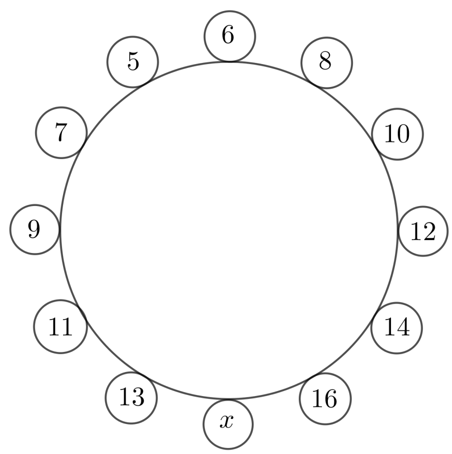
Therefore, when $a = 4$, the solution is $x = -5$.

Case 3: $a = 8$

Since the number on each card is different and we know that someone is holding a card with a 6, then the integer to right of the 8 must be 7, 9, or 10. Furthermore, since someone is already holding a 5 and someone is already holding a 6, every other integer to the the right of 8 must be 7 or greater.

The integer to the left of 5 is either 3, 4, or 7. If it is 3 or 4, then since someone is already holding the 5 and someone is already holding the 6, every integer to the left of 5 must be less than 5. Since x is both to the right of 8 and to the left of 5, if there is a 3 or a 4 to the left of 5, then x must be both 7 or greater and less than 5. This is not possible.

Therefore, if a solution exists when $a = 8$, then the integer to the left of 5 must be 7. The integer to the left of 7 must be 5, 6, 8, or 9. Since the 5, 6, and 8 are already placed, then the only possible integer to the left of 7 is 9. Similarly, the only possible integer to the right of 8 is 10. Thus, the integer to the left of 9 must be 11. Continuing in this manner, we get the table set up shown below.



From here, the only possible solution is $x = 15$.

Therefore, when $a = 8$, the solution is $x = 15$.

Therefore, the possible values for x are $x = -5$ or $x = 15$.

Data Management (D)

Take me to the cover

Problem of the Week
Problem E
A New Pair of Dice

A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is *fair*, which means that when it is rolled each of its faces has the same probability of being the top face. When two standard six-sided dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12.

Noemi creates two special six-sided dice that are also fair, but have non-standard numbers on their faces. Numbers on these special dice are positive integers and may appear more than once. The largest number on one of her special dice is 8. When the two special dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been used.

Determine all possible pairs of special dice that Noemi could have created.

Problem of the Week

Problem E and Solution

A New Pair of Dice

Problem

A standard six-sided die has its faces marked with the numbers 1, 2, 3, 4, 5, and 6. The die is *fair*, which means that when it is rolled each of its faces has the same probability of being the top face. When two standard six-sided dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12.

Noemi creates two special six-sided dice that are also fair, but have non-standard numbers on their faces. Numbers on these special dice are positive integers and may appear more than once. The largest number on one of her special dice is 8. When the two special dice are rolled and the numbers on the top faces are added together, the sums range from 2 to 12 and the probability of obtaining each sum is the same as it would be if two standard dice had been used.

Determine all possible pairs of special dice that Noemi could have created.

Solution

We first examine what happens when two standard dice are rolled. To do so, we create a table where the columns show the possible numbers on the top face for one die, the rows show the possible numbers on the top face for the other die, and each cell in the body of the table gives the sum of the corresponding numbers.

		Die 1					
		1	**2**	**3**	**4**	**5**	**6**
Die 2	**1**	2	3	4	5	6	7
	2	3	4	5	6	7	8
	3	4	5	6	7	8	9
	4	5	6	7	8	9	10
	5	6	7	8	9	10	11
	6	7	8	9	10	11	12

From this table we can determine the probability of each sum by counting the number of times each sum appears in the table, and dividing by 36, the total number of possible outcomes.

Sum	2	3	4	5	6	7	8	9	10	11	12
Probability	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

We need the pair of special dice to give these same probabilities when rolled together. Since the dice each have 6 sides, the total number of possible outcomes will still be $6 \times 6 = 36$.

We need the smallest possible sum to be 2. The only way to get a sum of 2 is if each die has a 1 on it. Thus, the smallest number on each die is 1. Furthermore, we need the probability that a sum of 2 is rolled to be $\frac{1}{36}$. Since there are 36 possible outcomes, this means that we need exactly 1 way to get a sum of 2. Therefore, exactly one face on each die must have a 1.

The largest possible sum must be 12 and we need the probability that this sum is rolled to be $\frac{1}{36}$. This means that we need exactly 1 way to get a sum of 12. Since the largest number on one die is 8, it follows that the other die must have exactly one 4. Furthermore, the die with the 8 must contain exactly one 8 and the die with the 4 must have exactly one 4 and no larger number.

Thus, we know that the numbers on one die, from least to greatest, must be (1, ___, ___, ___, ___, 4) and the numbers on the other die, from least to greatest, must be (1, ___, ___, ___, ___, 8), where ___ represents a number we still have to determine.

On the die where the largest number is 4, the remaining numbers must be all 2s and 3s. Since we need the probability that a sum of 3 is rolled to be $\frac{2}{36}$, it follows that there cannot be more than two 2s on this die, otherwise there would be more than 2 ways to get a sum of 3. Similarly, we need the probability that the sum of 11 is rolled to be $\frac{2}{36}$. Since $8 + 3 = 11$, it follows that there can not be more than two 3s on this die, otherwise there would be more than 2 ways to get a sum of 11.

Thus, on the die where the largest number is 4, we have concluded that there is exactly one 1, no more than two 2s, no more than two 3s, and exactly one 4. It follows that the numbers on that die must be (1, 2, 2, 3, 3, 4). We will now find possible numbers for the other die.

We need the probability that a sum of 4 is rolled to be $\frac{3}{36}$. This means that there must be exactly 3 ways to get a sum of 4. Currently we have 2 ways, so we need 1 more. If the die with the 8 has a 2 on a face, then there will be 2 more ways to get a sum of 4. Since we know that this die has exactly one 1, then it must have a 3 on it. Then we will have 3 ways to get a sum of 4, as desired. Thus, we know that the numbers on the second die must be (1, 3, ___, ___, ___, 8).

We need the probability that a sum of 10 is rolled to be $\frac{3}{36}$. That means that there must be exactly 3 ways to get a sum of 10. Currently we have 2 ways, so we need 1 more. If we had a 7 then we would have 2 more ways to get a sum of 10, which is too many. Thus, we must have one 6 on the die so that we can have 3 ways to get a sum of 10, as desired. Thus, we know that the numbers on the second die must be (1, 3, ___, ___, 6, 8).

We need the probability that a sum of 5 is rolled to be $\frac{4}{36}$. That means that there must be exactly 4 ways to get a sum of 5. Currently we have 3 ways, so we need one more. Thus, we must have one 4 on the die so that we can have 4 ways to get a sum of 5, as desired. Thus, we know that the numbers on the second die must be (1, 3, 4, ___, 6, 8).

Since we can't add another number that is currently on the die without changing the probability of rolling a sum we have already looked at, it follows that the remaining number must be a 5. Thus, it must be the case that the numbers on the first die are (1, 2, 2, 3, 3, 4) and the numbers on the second die are (1, 3, 4, 5, 6, 8).

We now need to check that this pair of dice satisfies the conditions of the problem. To do so, we create a table where the columns show the possible numbers on the top face for one die, the rows show the possible numbers on the top face for the other die, and each cell in the body of the table gives the sum of the corresponding numbers.

		Special Die 1					
		1	**2**	**2**	**3**	**3**	**4**
Special Die 2	**1**	2	3	3	4	4	5
	3	4	5	5	6	6	7
	4	5	6	6	7	7	8
	5	6	7	7	8	8	9
	6	7	8	8	9	9	10
	8	9	10	10	11	11	12

From this table we can determine the probability of each sum by counting the number of times each sum appears in the table, and dividing by 36, the total number of possible outcomes.

It turns out that these probabilities match the probabilities when two standard dice are rolled, so we can conclude that there is only one possible pair of special dice that Noemi could have created. The numbers on this pair of dice must be (1, 2, 2, 3, 3, 4) and (1, 3, 4, 5, 6, 8).

Geometry & Measurement (G)

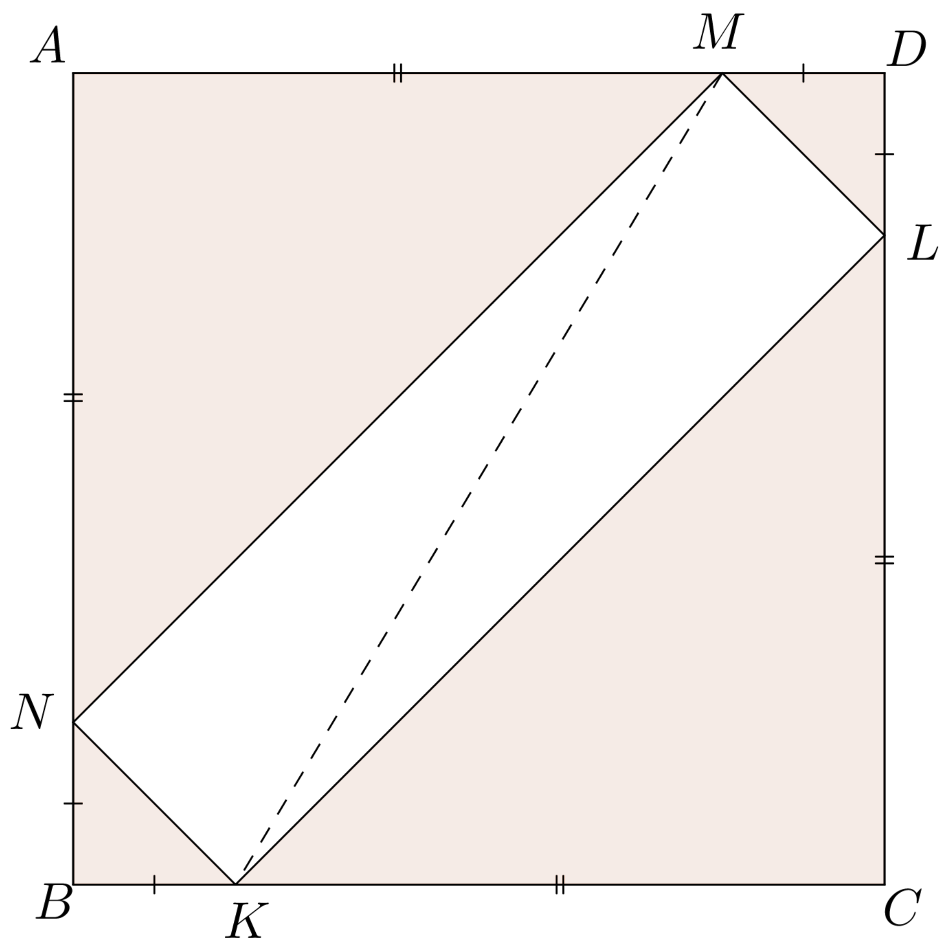
Take me to the cover

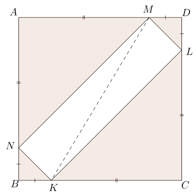
Problem of the Week

Problem E

Diagonal Distance

Square $ABCD$ has K on BC, L on DC, M on AD, and N on AB such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm^2, what is the length of MK?





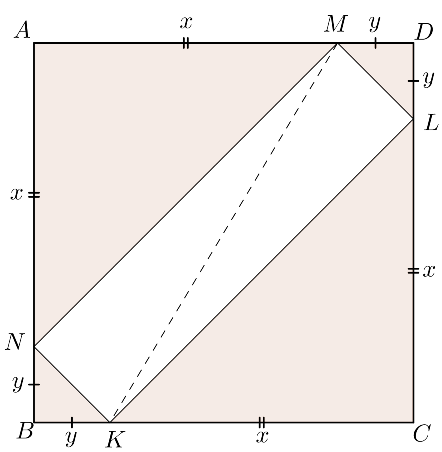
Problem of the Week
Problem E and Solution
Diagonal Distance

Problem

Square $ABCD$ has K on BC, L on DC, M on AD, and N on AB such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm^2, what is the length of MK?

Solution

Let x represent the lengths of the equal sides of $\triangle AMN$ and $\triangle LKC$, and let y represent the lengths of the equal sides of $\triangle MDL$ and $\triangle BNK$.



Thus, area $\triangle AMN =$ area $\triangle LKC = \frac{1}{2}x^2$, and area $\triangle MDL =$ area $\triangle BNK = \frac{1}{2}y^2$.

Therefore, the total area of the four triangles is equal to $\frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 = x^2 + y^2$. Since we're given that this area is 50 cm^2, we have $x^2 + y^2 = 50$.

Three different solutions to find the length of MK are provided.

Solution 1

In $\triangle AMN$, $MN^2 = AM^2 + AN^2 = x^2 + x^2$, and in $\triangle BNK$, $NK^2 = BN^2 + BK^2 = y^2 + y^2$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= x^2 + x^2 + y^2 + y^2 \\ &= x^2 + y^2 + x^2 + y^2 \\ &= 50 + 50 \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 2

In $\triangle AMN$, $MN^2 = x^2 + x^2 = 2x^2$. Therefore, $MN = \sqrt{2}x$, since $x > 0$.
In $\triangle BNK$, $NK^2 = y^2 + y^2 = 2y^2$. Therefore, $NK = \sqrt{2}y$, since $y > 0$.

Since MK is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

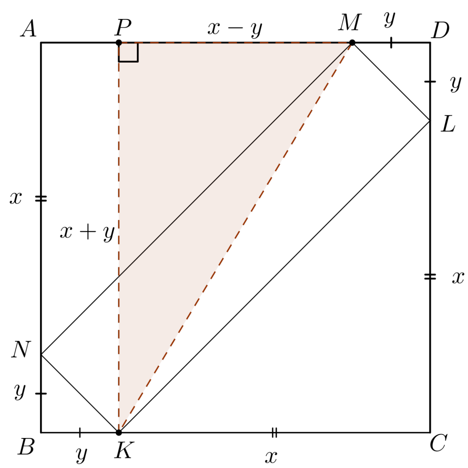
$$\begin{aligned} MK^2 &= MN^2 + NK^2 \\ &= (\sqrt{2}x)^2 + (\sqrt{2}y)^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 3

We construct the line segment KP, where P lies on AD such that KP is perpendicular to AD.

Then $APKB$ is a rectangle. Furthermore, $AP = BK = y$, $PK = AB = x + y$, and
$PM = AM - AP = x - y$.



Since $\triangle PKM$ is a right-angled triangle, by the Pythagorean Theorem we have

$$\begin{aligned} MK^2 &= PM^2 + PK^2 \\ &= (x - y)^2 + (x + y)^2 \\ &= x^2 - 2xy + y^2 + x^2 + 2xy + y^2 \\ &= 2x^2 + 2y^2 \\ &= 2(x^2 + y^2) \\ &= 2(50) \\ &= 100 \end{aligned}$$

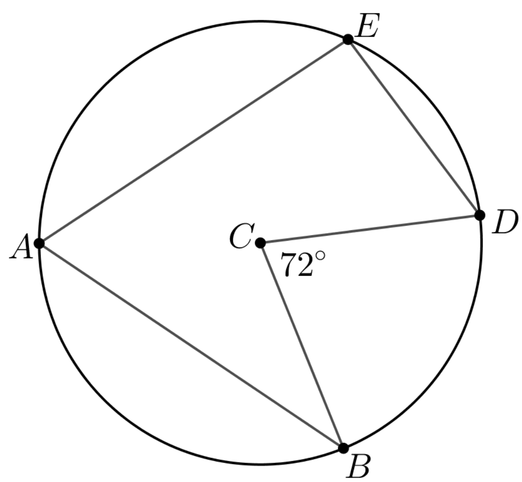
Since $MK > 0$, we have $MK = 10$ cm.

Problem of the Week

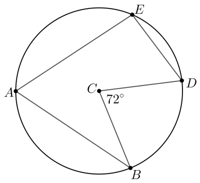
Problem E

Find Another Angle

The points A, B, D, and E lie on the circumference of a circle with centre C, as shown.



If $\angle BCD = 72^\circ$ and $CD = DE$, then determine the measure of $\angle BAE$.



Problem of the Week

Problem E and Solution

Find Another Angle

Problem

The points A, B, D, and E lie on the circumference of a circle with centre C, as shown.

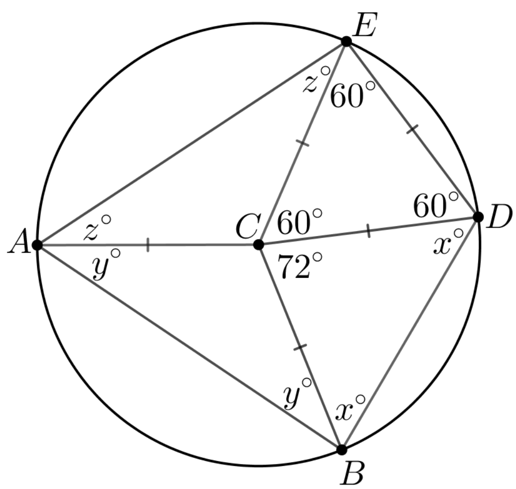
If $\angle BCD = 72^\circ$ and $CD = DE$, then determine the measure of $\angle BAE$.

Solution

We draw radii from C to points A and E on the circumference, and join B to D. Since CA, CB, CD, and CE are all radii, $CA = CB = CD = CE$.

We're given that $CD = DE$. Since $CD = CE$, we have $CD = CE = DE$, and thus $\triangle CDE$ is equilateral. It follows that $\angle ECD = \angle CED = \angle CDE = 60^\circ$.

Let $\angle CDB = x^\circ$, $\angle CBA = y^\circ$, and $\angle CAE = z^\circ$.
Since $CB = CD$, $\triangle CBD$ is isosceles. Therefore, $\angle CBD = \angle CDB = x^\circ$.
Since $CA = CB$, $\triangle CAB$ is isosceles. Therefore, $\angle CAB = \angle CBA = y^\circ$.
Since $CE = CA$, $\triangle CEA$ is isosceles. Therefore, $\angle CEA = \angle CAE = z^\circ$.



Since the angles in a triangle sum to 180°, from $\triangle CBD$ we have $x^\circ + x^\circ + 72^\circ = 180^\circ$. Thus, $2x^\circ = 108^\circ$ and $x = 54$.

Since $ABDE$ is a quadrilateral and the sum of the interior angles of a quadrilateral is equal to 360°, we have

$$\begin{aligned}
\angle BAE + \angle ABD + \angle BDE + \angle DEA &= 360^\circ \\
(y^\circ + z^\circ) + (y^\circ + x^\circ) + (x^\circ + 60^\circ) + (60^\circ + z^\circ) &= 360^\circ \\
2x + 2y + 2z &= 240 \\
x + y + z &= 120 \\
54 + y + z &= 120 \\
y + z &= 66
\end{aligned}$$

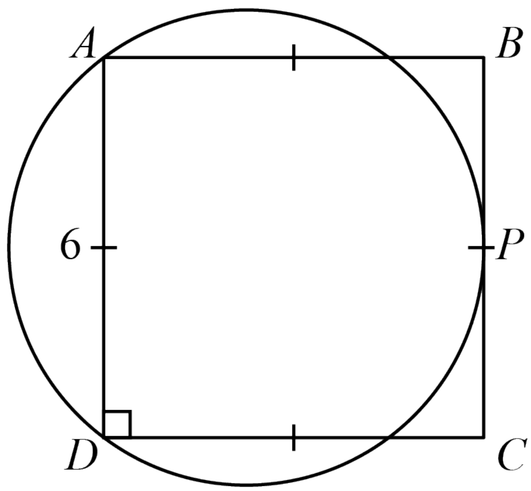
Since $\angle BAE = (y + z)^\circ$, then $\angle BAE = 66^\circ$.

Problem of the Week

Problem E

Shape Building

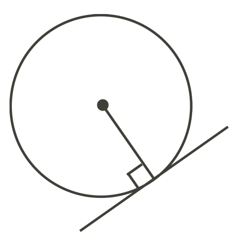
Sina drew square $ABCD$ with side length 6 cm on a piece of paper and passed the paper to Theo. Theo drew a circle on top of the square so that the circle passes through A and D, and the circle is tangent to side BC at point P.

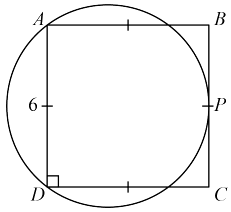


Determine the radius of the circle.

NOTE: You may find the following known result about circles useful:

If a line is tangent to a circle, then the perpendicular to that line at the point of tangency passes through the centre of the circle.





Problem of the Week
Problem E and Solution
Shape Building

Problem

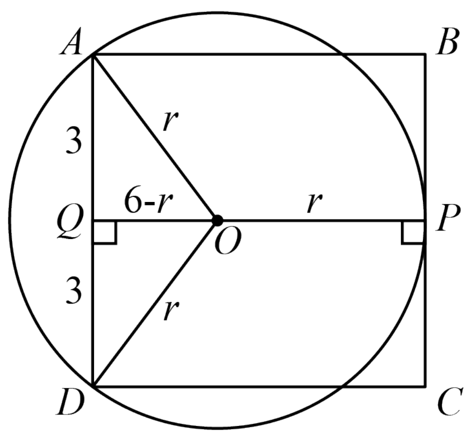
Sina drew square $ABCD$ with side length 6 cm on a piece of paper and passed the paper to Theo. Theo drew a circle on top of the square so that the circle passes through A and D, and the circle is tangent to side BC at point P. Determine the radius of the circle.

Solution

Let O be the centre of the circle and r be the radius. Construct line segment PQ perpendicular to CB with Q on side AD of the square. Since CB is tangent to the circle with point of tangency P, PQ must pass through the centre of the circle, O. Therefore, $PO = r$.

Since $PQ \perp BC$, $PQ \parallel AB$, and $PQ = AB = 6$, then $QO = PQ - PO = 6 - r$. Since A and D are on the circle, $AO = DO = r$.

Using the Pythagorean Theorem, $AQ^2 = AO^2 - QO^2 = r^2 - (6-r)^2$ and $DQ^2 = DO^2 - QO^2 = r^2 - (6-r)^2$. Therefore, $AQ^2 = DQ^2$ and $AQ = DQ$ follows. Since $AQ = DQ$ and $AQ + QD = AD = 6$, we can substitute to obtain $AQ + AQ = 2AQ = 6$ or $AQ = 3$.



Using the Pythagorean Theorem in $\triangle AQO$,

$$\begin{aligned} AO^2 &= AQ^2 + QO^2 \\ r^2 &= 3^2 + (6-r)^2 \\ r^2 &= 9 + 36 - 12r + r^2 \\ 12r &= 45 \\ r &= \frac{45}{12} \\ r &= 3.75 \end{aligned}$$

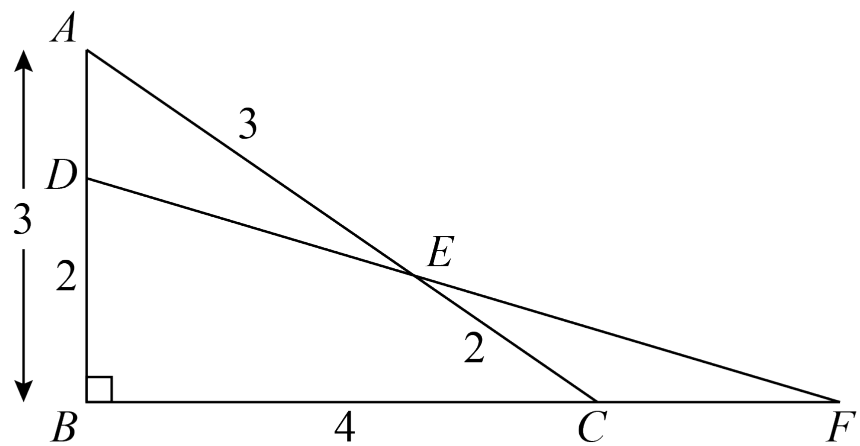
Therefore, the radius of the circle is 3.75 cm.

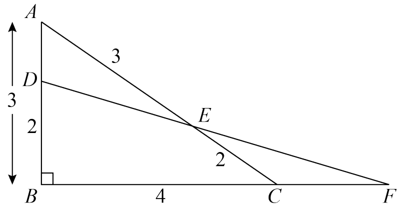
Problem of the Week

Problem E

Overlapping Shapes 3

Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90^\circ$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that D lies on AB, F lies on the extension of BC, $DB = 2$ cm, and sides AC and DF meet at E. If $AE = 3$ cm and $EC = 2$ cm, determine the length of CF.





Problem of the Week

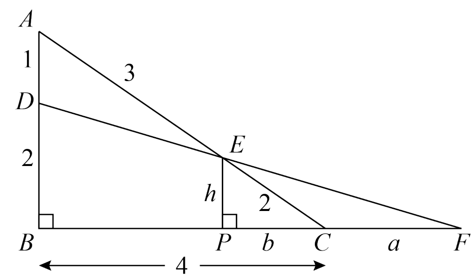
Problem E and Solution

Overlapping Shapes 3

Problem

Austin draws $\triangle ABC$ with $AB = 3$ cm, $BC = 4$ cm, and $\angle ABC = 90^\circ$. Lachlan then draws $\triangle DBF$ on top of $\triangle ABC$ so that D lies on AB, F lies on the extension of BC, $DB = 2$ cm, and sides AC and DF meet at E. If $AE = 3$ cm and $EC = 2$ cm, determine the length of CF.

Solution



Since $AB = 3$ and $DB = 2$, it follows that $AD = 1$ cm.
Draw a perpendicular from E to BF.
Let P be the point where the perpendicular intersects BF.
Let $CF = a$, $PC = b$, and $EP = h$.
We will now proceed with three solutions. The first two solutions depend on this setup. The first uses similar triangles, the second uses trigonometry, and the third uses coordinate geometry.

Solution 1

Since EP is perpendicular to BF, we know $\angle EPF = 90^\circ$. Also, $\angle ECP = \angle ACB$ (same angle). Therefore, $\triangle ABC \sim \triangle EPC$ (by angle-angle triangle similarity).

From the similarity, $\dfrac{AC}{BC} = \dfrac{EC}{PC}$, so $\dfrac{5}{4} = \dfrac{2}{b}$ or $b = \dfrac{8}{5}$. Also, $\dfrac{AC}{AB} = \dfrac{EC}{EP}$, so $\dfrac{5}{3} = \dfrac{2}{h}$ or $h = \dfrac{6}{5}$.

Now let's calculate PF. We know $\angle EPF = \angle DBF = 90^\circ$ and $\angle EFP = \angle DFB$ (same angle).

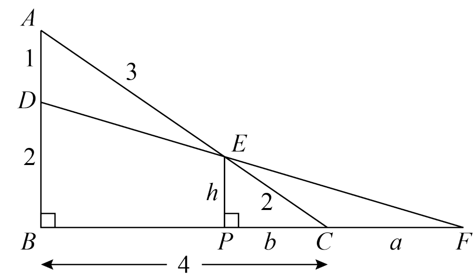
Therefore, $\triangle DBF \sim \triangle EPF$ (by angle-angle triangle similarity). This tells us $\dfrac{DB}{BF} = \dfrac{EP}{PF}$.

Since $BF = BC + CF = 4 + a$ and $PF = PC + CF = \frac{8}{5} + a$, we have

$$\begin{aligned}
\frac{DB}{BF} &= \frac{EP}{PF} \\
\frac{2}{4+a} &= \frac{\frac{6}{5}}{\frac{8}{5}+a} \\
\frac{16}{5} + 2a &= \frac{24}{5} + \frac{6}{5}a \\
2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5} \\
\frac{4}{5}a &= \frac{8}{5} \\
a &= 2
\end{aligned}$$

Therefore, $CF = 2$ cm.

Solution 2



In $\triangle EPC$, $\sin(\angle ECP) = \frac{h}{2}$. In $\triangle ABC$, $\sin(\angle ACB) = \frac{3}{5}$.
Since $\angle ECP = \angle ACB$ (same angle),

$$\begin{aligned}\sin(\angle ECP) &= \sin(\angle ACB)\\ \frac{h}{2} &= \frac{3}{5}\\ h &= \frac{6}{5}\end{aligned}$$

Since $\triangle EPC$ is a right-angled triangle,

$$\begin{aligned}EP^2 + PC^2 &= EC^2\\ h^2 + b^2 &= 2^2\\ \left(\frac{6}{5}\right)^2 + b^2 &= 4\\ b^2 &= 4 - \frac{36}{25}\\ b^2 &= \frac{64}{25}\\ b &= \frac{8}{5}, \quad \text{since} \quad b > 0\end{aligned}$$

In $\triangle EPF$, $\tan(\angle EFP) = \dfrac{EP}{PF} = \dfrac{h}{a+b} = \dfrac{\frac{6}{5}}{a+\frac{8}{5}}$.

In $\triangle DBF$, $\tan(\angle DFB) = \dfrac{DB}{BF} = \dfrac{2}{4+a}$.

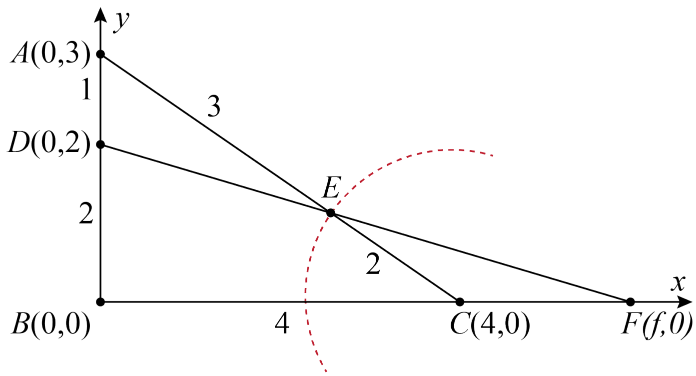
Since $\angle EFP = \angle DFB$ (same angle),

$$\begin{aligned}\tan(\angle EFP) &= \tan(\angle DFB)\\ \frac{\frac{6}{5}}{a+\frac{8}{5}} &= \frac{2}{4+a}\\ \frac{24}{5} + \frac{6}{5}a &= 2a + \frac{16}{5}\\ 2a - \frac{6}{5}a &= \frac{24}{5} - \frac{16}{5}\\ \frac{4}{5}a &= \frac{8}{5}\\ a &= 2\end{aligned}$$

Therefore, $CF = 2$ cm.

Solution 3

We will use coordinate geometry in this solution, and place B at the origin. Using the given information, D is at $(0, 2)$, A is at $(0, 3)$, C is at $(4, 0)$, and F is on the positive x-axis at $(f, 0)$ with $f > 4$. Consider the circle through E with centre $C(4, 0)$. Since $CE = 2$, the radius of this circle is 2. Thus, the equation of this circle is $(x - 4)^2 + y^2 = 4$.



The line passing through $A(0, 3)$ and $C(4, 0)$ has y-intercept 3 and slope $-\frac{3}{4}$, and so has equation $y = -\frac{3}{4}x + 3$. Since E lies on the line with equation $y = -\frac{3}{4}x + 3$ and the circle with equation $(x - 4)^2 + y^2 = 4$, to find the coordinates of E, we substitute $y = -\frac{3}{4}x + 3$ for y in $(x - 4)^2 + y^2 = 4$. Note that E is in the first quadrant so $x > 0$ and $y > 0$.

Doing so, we get

$$(x - 4)^2 + \left(-\frac{3}{4}x + 3\right)^2 = 4$$

Expanding the left side, we get

$$x^2 - 8x + 16 + \frac{9}{16}x^2 - \frac{9}{2}x + 9 = 4$$

Multiplying by 16, we get

$$16x^2 - 128x + 256 + 9x^2 - 72x + 144 = 64$$

Simplifying, we get

$$25x^2 - 200x + 336 = 0$$

Factoring, we then get

$$(5x - 12)(5x - 28) = 0$$

It follows that $x = \frac{12}{5}$ or $x = \frac{28}{5}$. Substituting $x = \frac{12}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = \frac{6}{5}$.

Substituting $x = \frac{28}{5}$ in $y = -\frac{3}{4}x + 3$, we obtain $y = -\frac{6}{5}$. But E is in the first quadrant so $y > 0$, and this second possibility is inadmissible. It follows that E has coordinates $\left(\frac{12}{5}, \frac{6}{5}\right)$.

We can now find the equation of the line containing $D(0, 2)$, $E\left(\frac{12}{5}, \frac{6}{5}\right)$, and $F(f, 0)$. This line has y-intercept 2, slope equal to $\dfrac{\frac{6}{5} - 2}{\frac{12}{5} - 0} = \dfrac{-\frac{4}{5}}{\frac{12}{5}} = -\frac{1}{3}$, and thus has equation $y = -\frac{1}{3}x + 2$.

The point $F(f, 0)$ lies on this line, so $0 = -\frac{1}{3}(f) + 2$, which leads to $f = 6$. Thus, the point F has coordinates $(6, 0)$. Since C is at $(4, 0)$ and F is at $(6, 0)$, $CF = 2$. It turns out that F also lies on the circle through E.

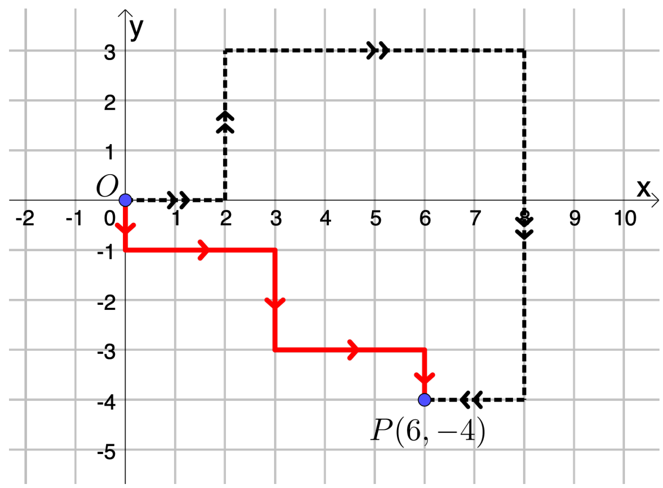
Therefore, $CF = 2$ cm.

Problem of the Week

Problem E

The Shortest Path

On the Cartesian plane, we draw grid lines at integer points along the x and y axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0,0)$ to $P(6,-4)$. One path has length 10 and the other has length 20.



There are many different paths along the grid lines from O to P, but the smallest possible length of such a path is 10. Let's call this smallest possible length the *path distance* from O to P.

Determine the number of points with integer coordinates for which the path distance from O to that point is 10.

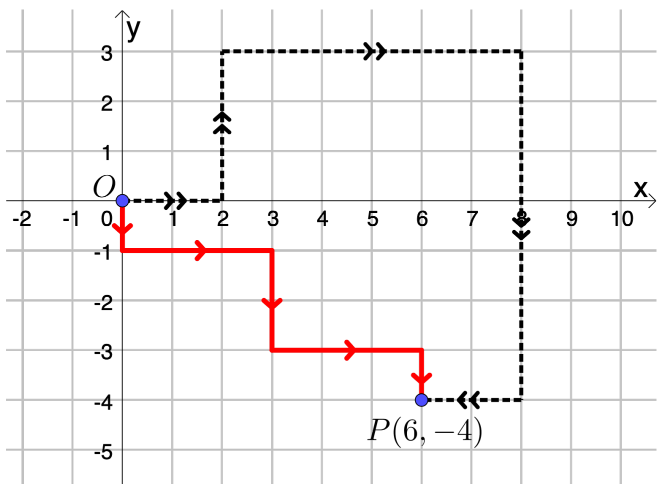
Problem of the Week

Problem E and Solution

The Shortest Path

Problem

On the Cartesian plane, we draw grid lines at integer points along the x and y axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0,0)$ to $P(6,-4)$. One path has length 10 and the other has length 20.



There are many different paths along the grid lines from O to P, but the smallest possible length of such a path is 10. Let's call this smallest possible length the *path distance* from O to P.

Determine the number of points with integer coordinates for which the path distance from O to that point is 10.

Solution

Solution 1

Let $Q(a,b)$ be a point that has path distance 10 from $O(0,0)$.

Let's first suppose that Q lies on the x or y axis.
The only point along the positive x-axis that has path distance 10 from the origin is $(10,0)$.
The only point along the negative x-axis that has path distance 10 from the origin is $(-10,0)$.
The only point along the positive y-axis that has path distance 10 from the origin is $(0,10)$.
The only point along the negative y-axis that has path distance 10 from the origin is $(0,-10)$.
Therefore, there are 4 points along the axes that have a path distance 10 from O.

Next, let's suppose $a>0$ and $b>0$, so Q is in the first quadrant.
Since the path distance from O to Q is 10, there must be a path from O to Q that moves a total of r units to the right and u units up (in some order) such that $r+u=10$. This means that Q is r units to the right of O and u units up from O. In other words, $a=r$ and $b=u$, so $a+b=r+u=10$.

The points (a, b) in the first quadrant that satisfy $a + b = 10$ where a and b are integers are $(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path distance 10 from O.

By symmetry, there are 9 points in each quadrant that have path distance 10 from O.
In quadrant 2, the points are $(-1, 9)$, $(-2, 8)$, $(-3, 7)$, $(-4, 6)$, $(-5, 5)$, $(-6, 4)$, $(-7, 3)$, $(-8, 2)$, $(-9, 1)$. In quadrant 3, the points are $(-1, -9)$, $(-2, -8)$, $(-3, -7)$, $(-4, -6)$, $(-5, -5)$, $(-6, -4)$, $(-7, -3)$, $(-8, -2)$, $(-9, -1)$. In quadrant 4, the points are $(1, -9)$, $(2, -8)$, $(3, -7)$, $(4, -6)$, $(5, -5)$, $(6, -4)$, $(7, -3)$, $(8, -2)$, $(9, -1)$.

Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have path distance 10 from O.

Solution 2

We are permitted 10 moves to get from the origin to a point by travelling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We examine the cases based on the number of horizontal moves.

- **0 horizontal moves**: Since there are 0 horizontal moves, there are 10 vertical moves. There are two possible endpoints, $(0, 10)$ and $(0, -10)$.
- **1 horizontal move**: Since there is 1 horizontal move, there are 9 vertical moves. There are four possible endpoints, $(-1, 9)$, $(-1, -9)$, $(1, 9)$, and $(1, -9)$.
- **2 horizontal moves**: Since there are 2 horizontal moves, there are 8 vertical moves. There are four possible endpoints, $(-2, 8)$, $(-2, -8)$, $(2, 8)$, and $(2, -8)$.
- **3 horizontal moves**: Since there are 3 horizontal moves, there are 7 vertical moves. There are four possible endpoints, $(-3, 7)$, $(-3, -7)$, $(3, 7)$, and $(3, -7)$.
- **4 horizontal moves**: Since there are 4 horizontal moves, there are 6 vertical moves. There are four possible endpoints, $(-4, 6)$, $(-4, -6)$, $(4, 6)$, and $(4, -6)$.
- **5 horizontal moves**: Since there are 5 horizontal moves, there are 5 vertical moves. There are four possible endpoints, $(-5, 5)$, $(-5, -5)$, $(5, 5)$, and $(5, -5)$.
- **6 horizontal moves**: Since there are 6 horizontal moves, there are 4 vertical moves. There are four possible endpoints, $(-6, 4)$, $(-6, -4)$, $(6, 4)$, and $(6, -4)$.
- **7 horizontal moves**: Since there are 7 horizontal moves, there are 3 vertical moves. There are four possible endpoints, $(-7, 3)$, $(-7, -3)$, $(7, 3)$, and $(7, -3)$.
- **8 horizontal moves**: Since there are 8 horizontal moves, there are 2 vertical moves. There are four possible endpoints, $(-8, 2)$, $(-8, -2)$, $(8, 2)$, and $(8, -2)$.
- **9 horizontal moves**: Since there are 9 horizontal moves, there is 1 vertical move. There are four possible endpoints, $(-9, 1)$, $(-9, -1)$, $(9, 1)$, and $(9, -1)$.
- **10 horizontal moves**: Since there are 10 horizontal moves, there are 0 vertical moves. There are two possible endpoints, $(-10, 0)$ and $(10, 0)$.

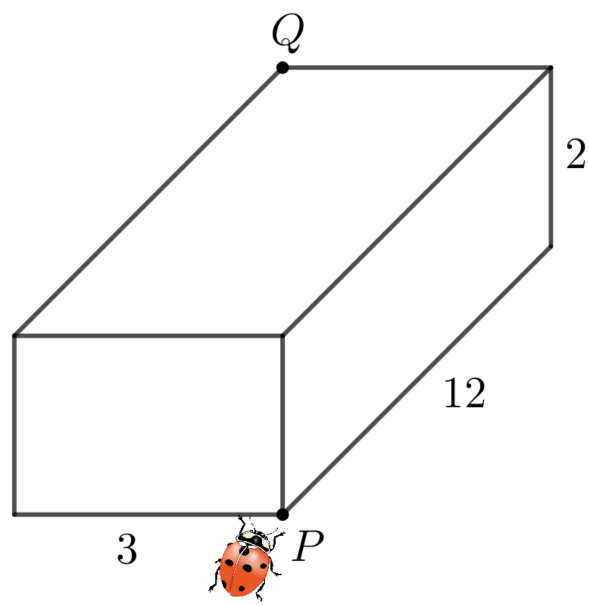
Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have path distance 10 from O.

Problem of the Week

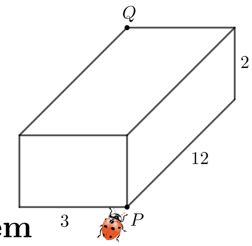
Problem E

Bug on the Outside

A ladybug walks on the surface of the 2 by 3 by 12 rectangular prism shown. The ladybug wishes to travel from P to Q.



What is the length of the shortest path from P to Q that the ladybug could take?



Problem of the Week
Problem E and Solution
Bug on the Outside

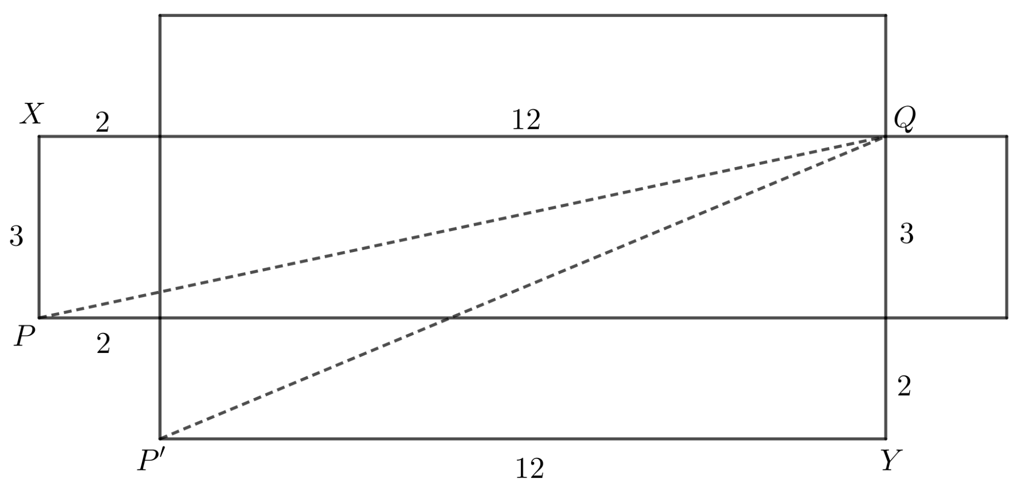
Problem

A ladybug walks on the surface of the 2 by 3 by 12 rectangular prism shown. The ladybug wishes to travel from P to Q.

What is the length of the shortest path from P to Q that the ladybug could take?

Solution

We fold out the sides of the prism so that they are laying on the same plane as the top of the prism. The diagram below shows the two-dimensional shape that results. As a result of folding out the sides, vertex P of the prism is a vertex of two different faces in the diagram. We call the second instance P'. We let X be the vertex adjacent to P along the side of length 3, and we let Y be the vertex adjacent to P' along the side of length 12.



The shortest distance for the ladybug to travel is in a straight line from P to Q or from P' to Q.

PQ is the hypotenuse of right-angled triangle PXQ. Using the Pythagorean Theorem,

$$PQ^2 = PX^2 + XQ^2 = 3^2 + 14^2 = 205$$

Thus, $PQ = \sqrt{205} \approx 14.3$, since $PQ > 0$.

$P'Q$ is the hypotenuse of right-angled triangle PYQ. Using the Pythagorean Theorem,

$$(P'Q)^2 = (P'Y)^2 + YQ^2 = 12^2 + 5^2 = 169$$

Thus, $P'Q = 13$, since $P'Q > 0$.

Since $P'Q < PQ$, the shortest distance for the ladybug to travel is 13 units on the surface of the block in a straight line from P' to Q.

Problem of the Week

Problem E

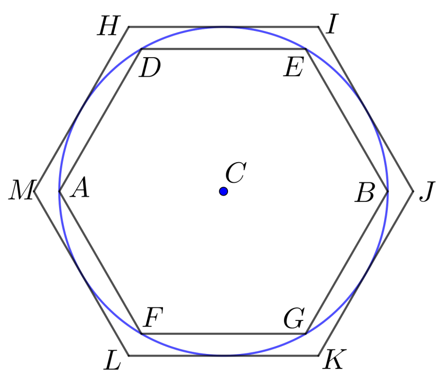
Pi Hexagons

Pi Day is an annual celebration of the mathematical constant π. Pi Day is observed on March 14, since 3, 1, and 4 are the first three significant digits of π.

Archimedes determined lower bounds for π by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. (An inscribed polygon of a circle has all of its vertices on the circle.) He also determined upper bounds for π by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. (A circumscribed polygon of a circle has all sides tangent to the circle. That is, each side of the polygon touches the circle in one spot.)

In this problem, we will determine a lower bound for π and an upper bound for π by considering an inscribed regular hexagon and a circumscribed regular hexagon in a circle of diameter 1.

Consider a circle with centre C and diameter 1. Since the circle has diameter 1, it has circumference equal to π. Now consider the inscribed regular hexagon $DEBGFA$ and the circumscribed regular hexagon $HIJKLM$.

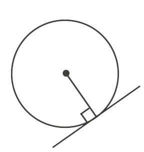


The perimeter of hexagon $DEBGFA$ will be less than the circumference of the circle, π, and will thus give us a lower bound for the value of π. The perimeter of hexagon $HIJKLM$ will be greater than the circumference of the circle, π, and will thus give us an upper bound for the value of π.

Using these hexagons, determine a lower and an upper bound for π.

NOTE: For this problem, you may want to use the following known results:

1. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.



2. For a circle with centre C, the centres of both the inscribed and circumscribed regular hexagons will be at C.

Problem of the Week

Problem E and Solution

Pi Hexagons

Problem

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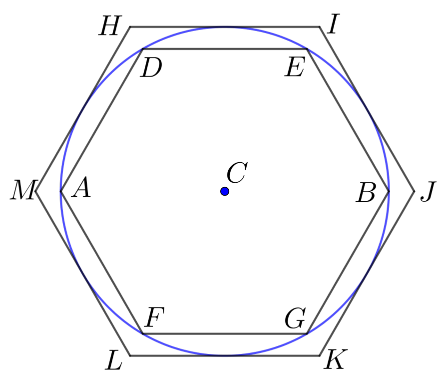

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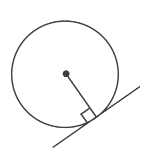
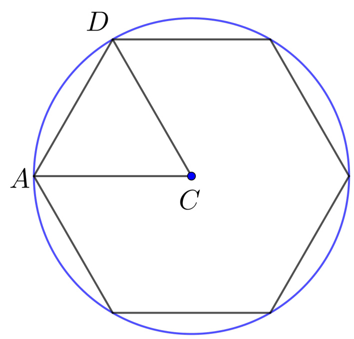


2. For a circle with centre C, the centres of both the inscribed and circumscribed regular hexagons will be at C.

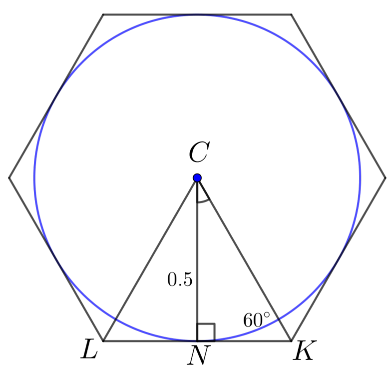
Solution

For the inscribed hexagon, draw line segments AC and DC, which are both radii of the circle.



Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre C, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6\left(\frac{1}{2}\right) = 3$. Since the perimeter of this hexagon is less than the circumference of the circle, this gives us a lower bound for π. That is, this tells us that $\pi > 3$.

For the circumscribed hexagon, draw line segments LC and KC. Since the circumscribed hexagon is a regular hexagon with centre C, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60^\circ$. Drop a perpendicular from C, meeting LK at N. We know that N must be the point of tangency. Thus, CN is a radius and so $CN = 0.5$. In $\triangle CNK$, $\angle NKC = \angle LKC = 60^\circ$.



Since $\angle CNK = 90^\circ$,

$$\begin{aligned}
\sin(\angle NKC) &= \frac{CN}{KC} \\
\sin(60^\circ) &= \frac{0.5}{KC} \\
\frac{\sqrt{3}}{2} &= \frac{0.5}{KC} \\
\sqrt{3}KC &= 1 \\
KC &= \frac{1}{\sqrt{3}}
\end{aligned}$$

But $\triangle LCK$ is equilateral, so $LK = KC = \frac{1}{\sqrt{3}}$.

Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$.

Since the perimeter of this hexagon is greater than the circumference of the circle, this gives us an upper bound for π. That is, this tells us that $\pi < \frac{6}{\sqrt{3}}$.

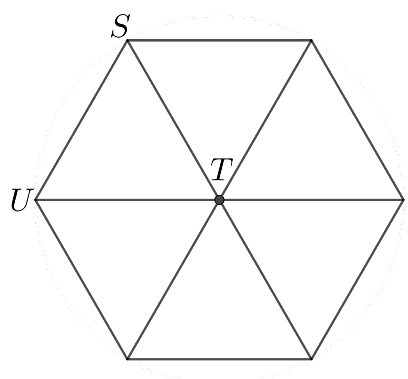
Therefore, the value for π is between 3 and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

Extension: Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on π. Can you?

Equilateral triangle justification:

In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre T. Draw line segments from T to each vertex and label two adjacent vertices S and U.



Since T is the centre of the hexagon, T is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360^\circ) = 60^\circ$.

Now consider $\triangle STU$. We know that $\angle STU = 60^\circ$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \angle TUS = \frac{180^\circ - 60^\circ}{2} = 60^\circ$.

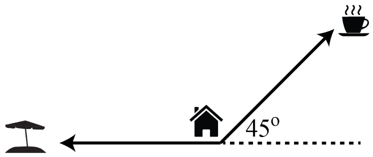
Therefore, all three angles in $\triangle STU$ are equal to 60° and so $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that all six triangles are all equilateral.

Problem of the Week

Problem E

Coffee Run

On Saturday morning at 8 a.m., Ayla and Hamza left their house. Ayla walked west towards the beach, and Hamza rode his e-scooter northeast to his favourite coffee shop, then headed to catch up with Ayla. Ayla walked at a constant speed of 4 km/h and Hamza rode his e-scooter at a constant speed of 20 km/h. If Hamza caught up with Ayla exactly 45 minutes after they left their house, what is the maximum possible distance between their house and the coffee shop?



Problem of the Week

Problem E and Solution

Coffee Run

Problem

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Solution

To find the maximum possible distance between their house and the coffee shop, we will assume that Hamza traveled in a straight line from his house to the coffee shop, and also from the coffee shop to catch up with Ayla. We will also assume that Hamza spent no time at the coffee shop. In reality these assumptions are unlikely, however they are necessary to determine the maximum possible distance.

Solution 1

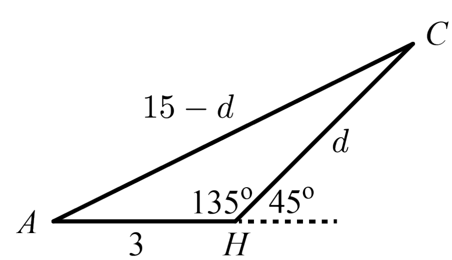
We know that the total time is 45 minutes, or $\frac{3}{4}$ of an hour, and in that time Ayla walked $\frac{3}{4} \times 4 = 3$ km.

Let t represent the time, in hours, that it took Hamza to travel to the coffee shop. Then, he took $\left(\frac{3}{4} - t\right)$ hours to travel from the coffee shop to meet up with Ayla. The distance between their house and the coffee shop is then $20t$ km, and the distance between the coffee shop and the point where Hamza met up with Ayla is $20\left(\frac{3}{4} - t\right) = (15 - 20t)$ km.

On the diagram, H represents their home, C represents the coffee shop, and A represents the point where Hamza met up with Ayla. We can determine that $\angle CHA = 180^\circ - 45^\circ = 135^\circ$. So $CH = 20t$ km, $AH = 3$ km, and $AC = (15 - 20t)$ km. If we let $d = 20t$, we can simplify CH to d and AC to $15 - d$.

Using the cosine law,

$$\begin{aligned}
AC^2 &= AH^2 + CH^2 - 2(AH)(CH)\cos(\angle CHA) \\
(15-d)^2 &= 3^2 + d^2 - 2(3)(d)\cos 135^\circ \\
225 - 30d + d^2 &= 9 + d^2 - 6d\left(-\frac{1}{\sqrt{2}}\right) \\
216 &= \frac{6d}{\sqrt{2}} + 30d \\
d &= 216 \div \left(\frac{6}{\sqrt{2}} + 30\right) \approx 6.3 \text{ km}
\end{aligned}$$



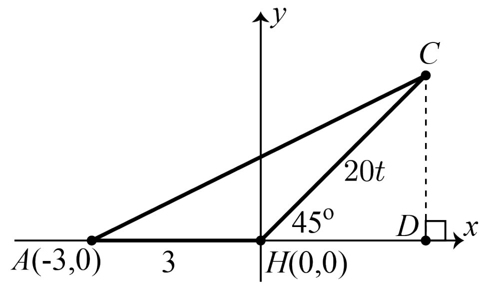
Therefore, the maximum possible distance between their house and the coffee shop is $216 \div \left(\frac{6}{\sqrt{2}} + 30\right) \approx 6.3$ km.

Solution 2

As in Solution 1, we know that the total time is 45 minutes, or $\frac{3}{4}$ of an hour, and in that time Ayla walked $\frac{3}{4} \times 4 = 3$ km. We will describe the positions of Ayla and Hamza in terms of points in the coordinate plane. Let the point $H(0,0)$ represent their home. Then Ayla walked along the negative x-axis, and the point at which she met up with Hamza is $A(-3,0)$.

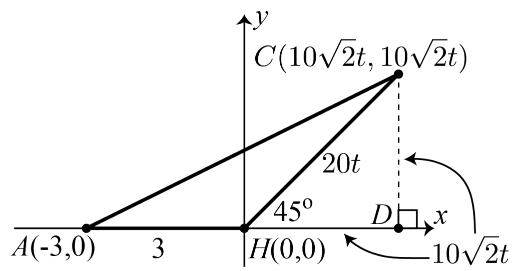
Let t represent the time, in hours, that it took Hamza to travel to the coffee shop, and let C represent the coffee shop. Then the length of CH is $20t$ km. To determine the coordinates of C, we will draw a vertical line from C that meets the x-axis at point D. Since $\angle CHD = 45°$, it follows that $\triangle CDH$ is an isosceles right-angled triangle.

$$\begin{aligned} \sin 45° &= \frac{CD}{20t} \\ \frac{1}{\sqrt{2}} &= \frac{CD}{20t} \\ CD &= \frac{20t}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{20\sqrt{2}t}{2} = 10\sqrt{2}t \end{aligned}$$



Since $\triangle CDH$ is an isosceles right-angled triangle, $HD = 10\sqrt{2}t$. Thus, the coordinates of C are $(10\sqrt{2}t, 10\sqrt{2}t)$. The distance between A and C, d_{AC}, can then be calculated.

$$\begin{aligned} (d_{AC})^2 &= (10\sqrt{2}t - (-3))^2 + (10\sqrt{2}t - 0)^2 \\ &= 100(2)t^2 + 60\sqrt{2}t + 9 + 100(2)t^2 \\ &= 400t^2 + 60\sqrt{2}t + 9 \\ d_{AC} &= \sqrt{400t^2 + 60\sqrt{2}t + 9} \end{aligned}$$



Since Hamza traveled at 20 km/h, the time it took him to travel this distance was $\frac{\sqrt{400t^2+60\sqrt{2}t+9}}{20}$ hours. It took Hamza $\frac{3}{4}$ of an hour to travel from H to C and then from C to A. Thus,

$$\begin{aligned} t + \frac{\sqrt{400t^2 + 60\sqrt{2}t + 9}}{20} &= \frac{3}{4} \\ 20t + \sqrt{400t^2 + 60\sqrt{2}t + 9} &= 15 \\ \sqrt{400t^2 + 60\sqrt{2}t + 9} &= 15 - 20t \\ 400t^2 + 60\sqrt{2}t + 9 &= (15 - 20t)^2 \\ 400t^2 + 60\sqrt{2}t + 9 &= 225 - 600t + 400t^2 \\ 600t + 60\sqrt{2}t &= 216 \\ t &= \frac{216}{600 + 60\sqrt{2}} \text{ hours} \end{aligned}$$

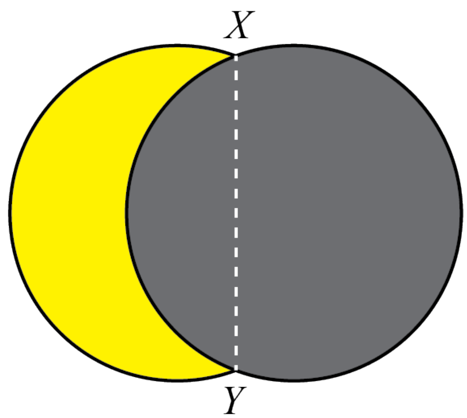
It follows that $HC = 20t = 20\left(\frac{216}{600+60\sqrt{2}}\right) = \frac{216}{30+3\sqrt{2}} \approx 6.3$ km. Therefore, the maximum possible distance between their house and the coffee shop is $\frac{216}{30+3\sqrt{2}} \approx 6.3$ km.

Problem of the Week

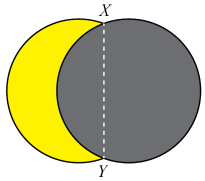
Problem E

Outside the Path of Totality

Yannick used a camera with a solar filter to capture a solar eclipse. From their location, the Moon blocked some, but not all of the Sun, so they saw a partial solar eclipse. When Yannick enlarged and printed the photo, they noticed that the distance represented by segment XY in the photo shown was 70 cm, and that the diameters of the Sun and the Moon were both 74 cm.



Determine the percentage of the Sun that is blocked by the Moon in Yannick's photo, rounded to 1 decimal place.



Problem of the Week
Problem E and Solution
Outside the Path of Totality

Problem

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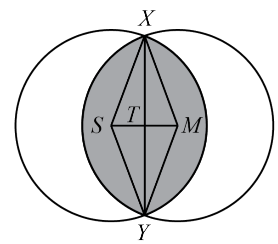
Solution

Let the centres of the Sun and Moon in the photo be S and M, respectively. We draw line segments SX, SY, MX, and MY. We draw line segment SM which intersects XY at T. The shaded region represents the portion of the Sun that is blocked by the Moon.

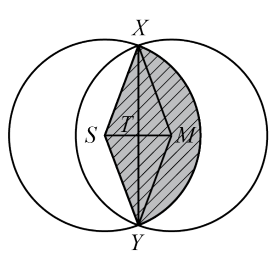


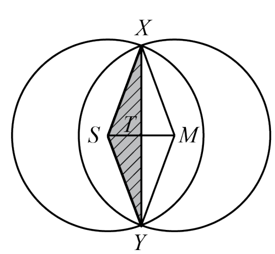
In our diagram we have drawn S and M inside the shaded region, but we need to prove they actually lie inside this region before we can proceed with the area calculation. Since each of the circles has radius $74 \div 2 = 37$, then $SX = SY = MX = MY = 37$. It follows that $\triangle SXY$ is isosceles. Similarly, $\triangle SXM$ and $\triangle SYM$ are congruent. Thus, $\angle XSM = \angle YSM$. Since $\triangle SXY$ is isosceles and SM bisects $\angle XSY$, then SM is perpendicular to XY at T, and $XT = TY = 70 \div 2 = 35$. By the Pythagorean Theorem in $\triangle STX$,

$$ST = \sqrt{SX^2 - XT^2} = \sqrt{37^2 - 35^2} = 12$$

Thus, $SM = 2 \times ST = 2 \times 12 = 24$. Since SM is smaller than the radius of the circles, it follows that S and M must lie inside the shaded region.

Now we can proceed with the area calculation. By symmetry, the area of the shaded region on each side of XY will be the same. The area of the shaded region on the right side of XY equals the area of acute sector XSY of the left circle minus the area of $\triangle SXY$. These areas are striped in the following diagrams.





First we find the area of $\triangle SXY$, which is $\frac{1}{2}(XY)(ST) = \frac{1}{2}(70)(12) = 420$.

Next we find the area of sector XSY. Using the cosine law in $\triangle SXY$,

$$\begin{aligned} XY^2 &= SX^2 + SY^2 - 2(SX)(SY)\cos(\angle XSY) \\ 70^2 &= 37^2 + 37^2 - 2(37)(37)\cos(\angle XSY) \\ 2162 &= -2738\cos(\angle XSY) \\ \angle XSY &= \cos^{-1}\left(-\frac{2162}{2738}\right) \approx 142.15^\circ \end{aligned}$$

Thus, the area of sector XSY is equal to $\frac{142.15^\circ}{360^\circ}\pi(37)^2$.

Then the area of the shaded region is equal to $2\left(\frac{142.15^\circ}{360^\circ}\pi(37)^2 - 420\right)$.

Finally, we calculate this area as a percentage of the area of the entire circle to obtain the following:

$$\frac{2\left(\frac{142.15^\circ}{360^\circ}\pi(37)^2 - 420\right)}{\pi(37)^2} \approx 59.4\%$$

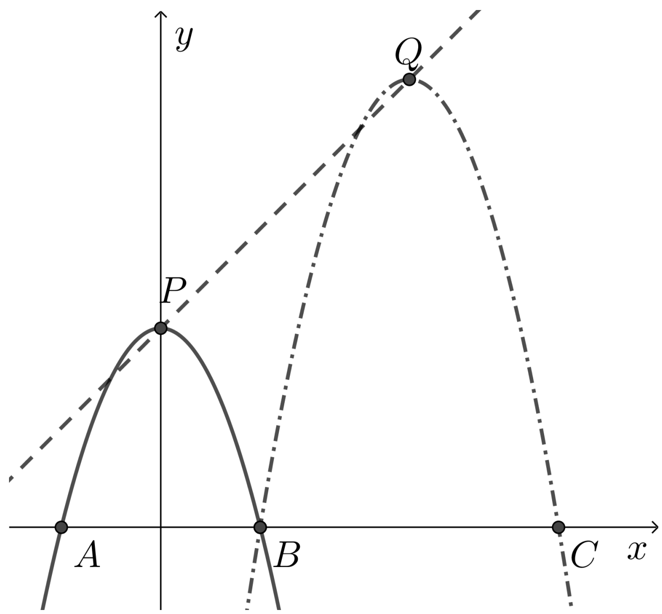
It follows that the percentage of the Sun that is blocked by the Moon in Yannick's photo is equal to approximately 59.4%.

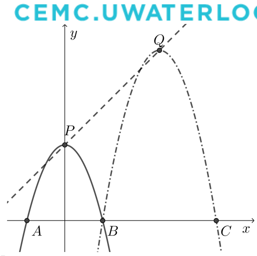
Problem of the Week
Problem E
Sliding Parabola

Suppose the parabola with equation $y = 4 - x^2$ has vertex at P and crosses the x-axis at points A and B, with B lying to the right of A on the x-axis.

This parabola is translated so that its vertex moves along the line $y = x + 4$ to the point Q. The new parabola crosses the x-axis at points B and C, with C lying to the right of B on the x-axis.

Determine the coordinates of C.





Problem of the Week
Problem E and Solution
Sliding Parabola

Problem

Suppose the parabola with equation $y = 4 - x^2$ has vertex at P and crosses the x-axis at points A and B, with B lying to the right of A on the x-axis.

This parabola is translated so that its vertex moves along the line $y = x + 4$ to the point Q. The new parabola crosses the x-axis at points B and C, with C lying to the right of B on the x-axis.

Determine the coordinates of C.

Solution

For the original parabola $y = -x^2 + 4$, the vertex is $P(0, 4)$ and the x-intercepts are $A(-2, 0)$ and $B(2, 0)$.

Let the vertex of the translated parabola be $Q(q, p)$. Since the new parabola is a translation of the original, the equation of this new parabola is $y = -(x - q)^2 + p$.

Since Q lies on the line $y = x + 4$, we have $p = q + 4$ and the equation of the new parabola is $y = -(x - q)^2 + q + 4$.

Since $B(2, 0)$ lies on the new parabola, we can substitute $(2, 0)$ into this equation:

$$\begin{aligned}
0 &= -(2 - q)^2 + q + 4 \\
0 &= -(q^2 - 4q + 4) + q + 4 \\
0 &= -q^2 + 5q \\
0 &= -q(q - 5)
\end{aligned}$$

Therefore, $q = 0$ or $q = 5$. The value $q = 0$ corresponds to point $P(0, 4)$ in the original parabola. Therefore, $q = 5$. From here we will show two solutions.

Solution 1

Since $q = 5$, the axis of symmetry for the new parabola is $x = 5$. To find C we need to reflect the point $B(2, 0)$ in the axis of symmetry to get $C(8, 0)$.

Solution 2

Since $q = 5$, then the vertex of the new parabola is $(5, 9)$ and the equation of this parabola is $y = -(x - 5)^2 + 9$.

Since C is an x-intercept of this parabola, to determine C we set $y = 0$ in the equation for the parabola and solve for x.

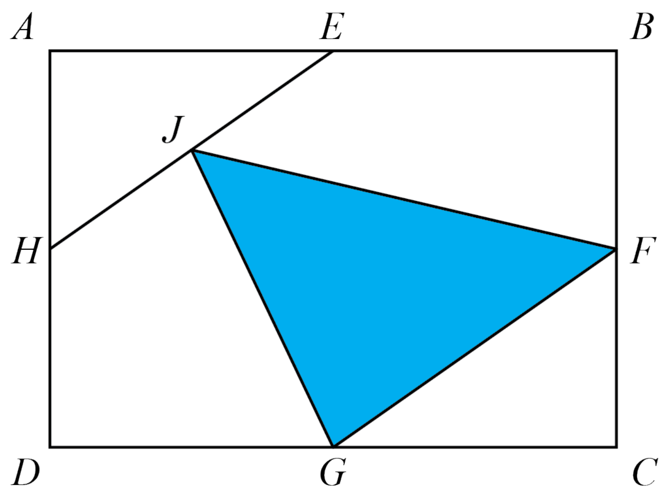
$$\begin{aligned}
0 &= -(x - 5)^2 + 9 \\
(x - 5)^2 &= 9 \\
x - 5 &= \pm 3 \\
x &= 8, 2
\end{aligned}$$

The value $x = 2$ corresponds to point B, and the value $x = 8$ corresponds to point C. Therefore, the coordinates of C are $(8, 0)$.

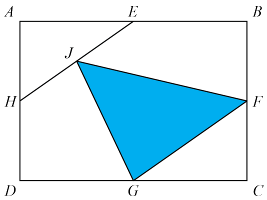
Problem of the Week
Problem E
Stained Glass

A stained glass window hanging is in the shape of a rectangle with a length of 8 cm and a width of 6 cm.

Rectangle $ABCD$ represents the window hanging with $AB = 8$ and $BC = 6$. The points E, F, G, and H are the midpoints of sides AB, BC, CD, and AD, respectively. The point J is the midpoint of line segment EH. Triangle FGJ is coloured blue.



Determine the area of the blue triangle.



Problem of the Week

Problem E and Solution

Stained Glass

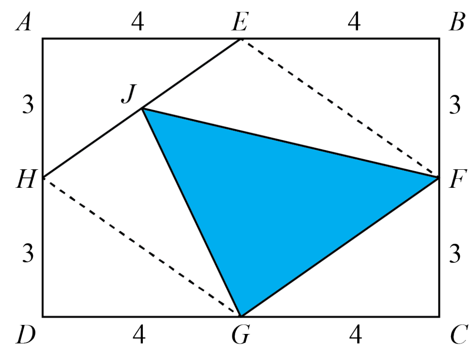
Problem

A stained glass window hanging is in the shape of a rectangle with a length of 8 cm and a width of 6 cm.

Rectangle $ABCD$ represents the window hanging with $AB = 8$ and $BC = 6$. The points E, F, G, and H are the midpoints of sides AB, BC, CD, and AD, respectively. The point J is the midpoint of line segment EH. Triangle FGJ is coloured blue. Determine the area of the blue triangle.

Solution

Solution 1



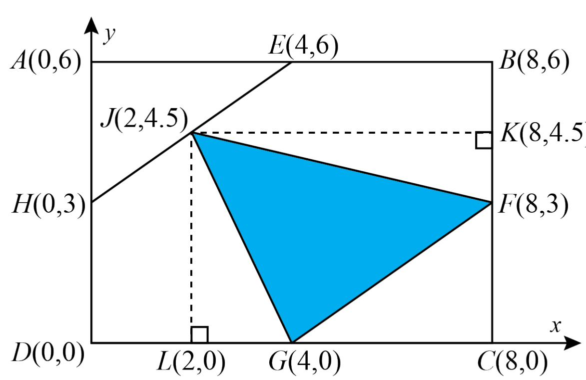
Since $ABCD$ is a rectangle and $AB = 8$, it follows that $AE = EB = DG = GC = 4$. Similarly, since $BC = 6$, it follows that $BF = FC = AH = HD = 3$.
Consider the four corner triangles, $\triangle HAE$, $\triangle EBF$, $\triangle FCG$, and $\triangle GDH$. Each of these triangles is a right-angled triangle with base 4 and height 3. Therefore, the total area of these four triangles is equal to $4 \times \frac{4\times 3}{2} = 24$.

The length of the hypotenuse of each of the four corner triangles is equal to $\sqrt{3^2 + 4^2} = 5$. Thus, $EF = FG = GH = EH = 5$, so $EFGH$ is a rhombus. Thus $EH \parallel FG$. The area of rhombus $EFGH$ is equal to the area of rectangle $ABCD$ minus the area of the four corner triangles. Thus, the area of rhombus $EFGH$ is $8 \times 6 - 24 = 24$.

Let h be the perpendicular distance between FG and EH. Then the area of rhombus $EFGH$ is $h \times FG$. Thus, $h \times 5 = 24$.

Triangle FGJ has base FG and height h, so its area is equal to $\frac{h\times FG}{2} = \frac{h\times 5}{2} = \frac{24}{2} = 12 \text{ cm}^2$.

Solution 2



In this solution we will use analytic geometry and set the coordinates of D to $(0, 0)$. Then $A(0, 6)$, $B(8, 6)$, and $C(8, 0)$ are the other corners of the rectangle. The midpoints E, F, G, and H have coordinates $(4, 6)$, $(8, 3)$, $(4, 0)$, and $(0, 3)$, respectively. Then J has coordinates $(2, 4.5)$. Let K have coordinates $(8, 4.5)$, and L have coordinates $(2, 0)$. Then $JKCL$ is a rectangle.

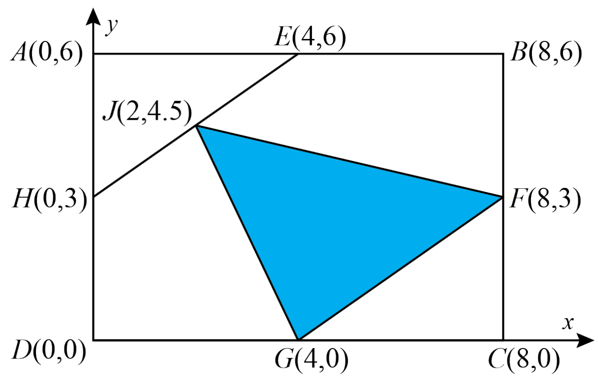
We can then calculate the area of $\triangle FGJ$ as follows.

$$\begin{aligned}\text{Area } \triangle FGJ &= \text{Area } JKCL - \text{Area } \triangle JKF - \text{Area } \triangle FCG - \text{Area } \triangle GLJ \\ &= JK \times CK - \frac{JK \times KF}{2} - \frac{FC \times CG}{2} - \frac{GL \times LJ}{2} \\ &= 6 \times 4.5 - \frac{6 \times 1.5}{2} - \frac{3 \times 4}{2} - \frac{2 \times 4.5}{2} \\ &= 27 - 4.5 - 6 - 4.5 \\ &= 12\end{aligned}$$

Therefore, the area of $\triangle FGJ$ is equal to 12 cm^2.

Solution 3

This solution also uses analytic geometry. As in Solution 2, set the coordinates of D to $(0, 0)$. Then $A(0, 6)$, $B(8, 6)$, and $C(8, 0)$ are the other corners of the rectangle. The midpoints E, F, G, and H have coordinates $(4, 6)$, $(8, 3)$, $(4, 0)$, and $(0, 3)$, respectively. Then J has coordinates $(2, 4.5)$.



The base of $\triangle FGJ$ is equal to the length of FG. Since $\triangle FCG$ is a right-angled triangle, $FG = \sqrt{CF^2 + CG^2} = \sqrt{3^2 + 4^2} = 5$. Line segments EH and FG each have a slope of $\frac{3}{4}$, so it follows that they are parallel. Thus, the height of $\triangle FGJ$ is equal to the perpendicular distance between EH and FG.

The line passing through F and G has slope $\frac{3}{4}$. The line perpendicular to FG, passing through G has slope $-\frac{4}{3}$ and y-intercept $\frac{16}{3}$. Therefore, its equation is $y = -\frac{4}{3}x + \frac{16}{3}$.

The line passing through EH has slope $\frac{3}{4}$ and y-intercept 3. Therefore, its equation is $y = \frac{3}{4}x + 3$. We can then determine the point of intersection of $y = \frac{3}{4}x + 3$ and $y = -\frac{4}{3}x + \frac{16}{3}$ by setting $\frac{3}{4}x + 3 = -\frac{4}{3}x + \frac{16}{3}$.

We multiply both sides of this equation by 12 and solve for x:

$$\begin{aligned}9x + 36 &= -16x + 64 \\ 25x &= 28 \\ x &= \frac{28}{25}\end{aligned}$$

The y-coordinate for this intersection point is then $y = \frac{3}{4}\left(\frac{28}{25}\right) + 3 = \frac{21}{25} + 3 = \frac{96}{25}$.

Then, the height of $\triangle FGJ$ is equal to the distance between $\left(\frac{28}{25}, \frac{96}{25}\right)$ and $G(4, 0)$, which is

$$\sqrt{\left(\frac{96}{25} - 0\right)^2 + \left(\frac{28}{25} - 4\right)^2} = \sqrt{\frac{9216}{625} + \frac{5184}{625}} = \sqrt{\frac{14\,400}{625}} = \sqrt{\frac{576}{25}} = \frac{24}{5}$$

Therefore, the area of $\triangle FGJ$ is equal to $\frac{1}{2} \times 5 \times \frac{24}{5} = 12$ cm^2.

Extension: Suppose $AB = p$ and $BC = q$, for some real numbers p and q. Determine the area of $\triangle FGJ$.

Number Sense (N)

Take me to the cover

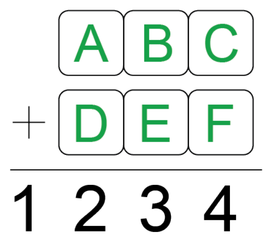
Problem of the Week

Problem E

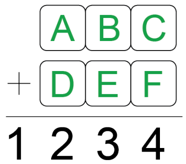
Arranging Tiles 3

Eliana has a box of tiles, each with an integer from 0 to 9 on it. Each integer appears on at least three tiles. Eliana creates larger numbers by placing tiles side by side. For example, using the tiles 3 and 7, Eliana can create the 2-digit number 37 or 73.

Using six of her tiles, Eliana forms two 3-digit numbers, ABC and DEF, that add to 1234.



Eliana then notices that $A > D$, $B > E$, and $C > F$. How many possible 6-tuples $(A,\ B,\ C,\ D,\ E,\ F)$ could she have chosen?



Problem of the Week
Problem E and Solution
Arranging Tiles 3

Problem

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Solution

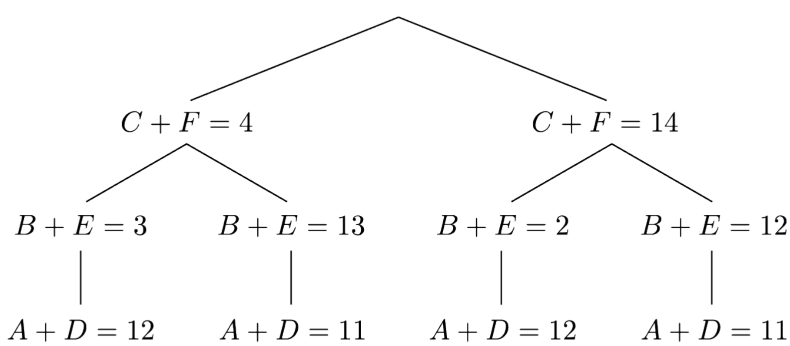
Since $C + F$ ends in a 4, then $C + F = 4$ or $C + F = 14$. The value of $C + F$ cannot be 20 or more, because C and F are digits. In the case that $C + F = 14$, we "carry" a 1 to the tens column. Now we will look at the tens column for these two cases.

- **Case 1:** $C + F = 4$
  Since the result in the tens column is 3 and there was no "carry" from the units column, it follows that $B + E$ ends in a 3. Then $B + E = 3$ or $B + E = 13$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 13$, we "carry" a 1 to the hundreds column.

- **Case 2:** $C + F = 14$
  Since the result in the tens column is 3 and there was a "carry" from the units column, it follows that $1 + B + E$ ends in a 3, so $B + E$ ends in a 2. Then $B + E = 2$ or $B + E = 12$. The value of $B + E$ cannot be 20 or more, because B and E are digits. In the case that $B + E = 12$, we "carry" a 1 to the hundreds column.

Since the result in the hundreds column is 12, then $A + D = 12$, or in the case when there was a "carry" from the tens column, $1 + A + D = 12$, so $A + D = 11$.

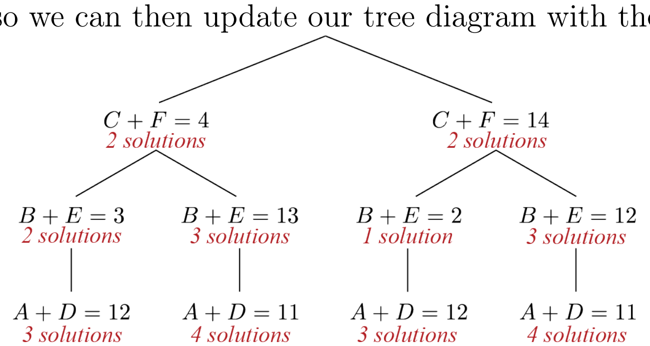
We summarize this information in the following tree.



We now look at the different possibilities for digits A, B, C, D, E, and F for each individual sum, with the restriction that $A > D$, $B > E$, and $C > F$.

Sum	Solutions	Number of Solutions
$C+F=4$	$C=4,\ F=0$ $C=3,\ F=1$	2
$C+F=14$	$C=9,\ F=5$ $C=8,\ F=6$	2
$B+E=2$	$B=2,\ E=0$	1
$B+E=3$	$B=3,\ E=0$ $B=2,\ E=1$	2
$B+E=12$	$B=9,\ E=3$ $B=8,\ E=4$ $B=7,\ E=5$	3
$B+E=13$	$B=9,\ E=4$ $B=8,\ E=5$ $B=7,\ E=6$	3
$A+D=11$	$A=9,\ D=2$ $A=8,\ D=3$ $A=7,\ D=4$ $A=6,\ D=5$	4
$A+D=12$	$A=9,\ D=3$ $A=8,\ D=4$ $A=7,\ D=5$	3

Since we can use each number at least 3 times, all combinations of solutions outlined in the table are possible, and so we can then update our tree diagram with the total number of solutions for each sum.



The first (leftmost) path through the tree corresponds to the sums $C+F=4$ (2 solutions), $B+E=3$ (2 solutions), and $A+D=12$ (3 solutions). Since there are 2 ways to achieve the first sum, and for each of these possibilities there are 2 ways to achieve the second sum, and for each of these possibilities there are 3 ways to achieve the third sum, the number of 6-tuples this path corresponds to is equal to $2 \times 2 \times 3 = 12$.

Similarly, the second path through the tree corresponds to the sums $C+F=4$ (2 solutions), $B+E=13$ (3 solutions), and $A+D=11$ (4 solutions). So the number of 6-tuples this path corresponds to is equal to $2 \times 3 \times 4 = 24$.

The third path through the tree corresponds to the sums $C+F=14$ (2 solutions), $B+E=2$ (1 solution), and $A+D=12$ (3 solutions). So the number of 6-tuples this path corresponds to is equal to $2 \times 1 \times 3 = 6$.

The fourth path through the tree corresponds to the sums $C+F=14$ (2 solutions), $B+E=12$ (3 solutions), and $A+D=11$ (4 solutions). So the number of 6-tuples this path corresponds to is equal to $2 \times 3 \times 4 = 24$.

Therefore, the total number of 6-tuples is $12+24+6+24=66$.

Problem of the Week

Problem E

Let's Paint

Painters R Us has been given a large painting job. Initially, Jim started painting by himself. In 15 days, working 9 hours each day, he was able to complete $\frac{3}{8}$ of the job. He decided to have Wanda join him for the remaining part of the job. Together they completed the job in another 10 days, each working 9 hours per day. If Wanda had originally done the job by herself, how many hours would it have taken her to finish the complete job?

Problem of the Week
Problem E and Solution
Let's Paint

Problem

Painters R Us has been given a large painting job. Initially, Jim started painting by himself. In 15 days, working 9 hours each day, he was able to complete $\frac{3}{8}$ of the job. He decided to have Wanda join him for the remaining part of the job. Together they completed the job in another 10 days, each working 9 hours per day. If Wanda had originally done the job by herself, how many hours would it have taken her to finish the complete job?

Solution

We must make some reasonable assumptions. We will assume that each painter worked at a constant rate each hour, every day. These rates may or may not have been the same for the two painters.

Since Jim completed $\frac{3}{8}$ of the job in 15 days, he would complete $\frac{1}{3}$ of $\frac{3}{8}$, or $\frac{1}{8}$, of the job in 5 days.

Since Jim had completed $\frac{3}{8}$ of the job when Wanda started to work, $\frac{5}{8}$ of the job was left to be completed. Together they completed $\frac{5}{8}$ of the job in 10 days. Since Jim can complete $\frac{1}{8}$ of the job in 5 days, he would have completed $\frac{2}{8}$ of the job in these 10 days. Therefore, Wanda completed the remaining $\frac{5}{8} - \frac{2}{8} = \frac{3}{8}$ of the job in these 10 days.

Since Wanda worked 9 hours a day, this means she completed $\frac{3}{8}$ of the job in $10 \times 9 = 90$ hours. Therefore, she completed $\frac{1}{8}$ of the job in 30 hours. Therefore, she could have completed the entire job on her own in $8 \times 30 = 240$ hours.

For Your Information:

Jim completed $\frac{1}{8}$ of the job in 5 days. The whole job could be completed by Jim in $8 \times 5 = 40$ days or 360 hours.

As it was, Jim worked a total of 25 days at 9 hours per day and Wanda worked 10 days at 9 hours per day. They worked a total of $25 \times 9 + 10 \times 9 = 315$ hours.

We know that together Jim and Wanda completed $\frac{5}{8}$ of the job in 10 days. Then, in 2 days they would have completed $\frac{1}{8}$ of the job and in 16 days they would have completed the entire job. That is, working together from the start they would have completed the job in $16 \times 9 = 144$ hours.

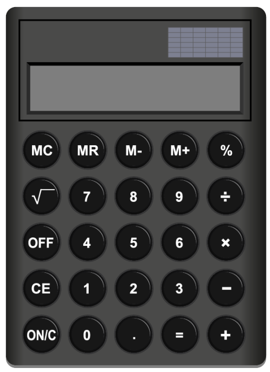
Problem of the Week

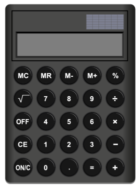
Problem E

Number Crunching

While waiting for the bus one day, Leo divided numbers on his calculator. He noticed that when 44 000 is divided by 18, the remainder is 8. He then noticed that the remainder is also 8 when 44 000 is divided by 24, and also when 44 000 is divided by 39. Leo then set out to find other numbers that had the same remainder (not necessarily 8), when divided by 18, 24, and 39.

How many five-digit positive integers have the same remainder when divided by 18, 24, and 39?





Problem of the Week

Problem E and Solution

Number Crunching

Problem

While waiting for the bus one day, Leo divided numbers on his calculator. He noticed that when 44 000 is divided by 18, the remainder is 8. He then noticed that the remainder is also 8 when 44 000 is divided by 24, and also when 44 000 is divided by 39. Leo then set out to find other numbers that had the same remainder (not necessarily 8), when divided by 18, 24, and 39. How many five-digit positive integers have the same remainder when divided by 18, 24, and 39?

Solution

Since $18 = 2 \times 3 \times 3$, $24 = 2 \times 2 \times 2 \times 3$, and $39 = 3 \times 13$, the lowest common multiple (LCM) of 18, 24, and 39 is $\text{LCM}(18, 24, 39) = 2 \times 2 \times 2 \times 3 \times 3 \times 13 = 936$.

Suppose n is a positive integer. Then the following statements are true:

Every integer of the form $936n$ will have a remainder of 0 when divided by 18, 24, and 39.
Every integer of the form $936n + 1$ will have a remainder of 1 when divided by 18, 24, and 39.
Every integer of the form $936n + 2$ will have a remainder of 2 when divided by 18, 24, and 39.
Every integer of the form $936n + 3$ will have a remainder of 3 when divided by 18, 24, and 39.

$$\vdots$$

Every integer of the form $936n + 16$ will have a remainder of 16 when divided by 18, 24, and 39.
Every integer of the form $936n + 17$ will have a remainder of 17 when divided by 18, 24, and 39.

However, every integer of the form $936n + 18$ will not have the same remainder when divided by 18, 24, and 39. The remainders will be 0, 18, and 18, respectively. Therefore, we need to find the number of five-digit integers that have the form $936n + r$ where $0 \le r \le 17$.

The smallest five-digit integer that is a multiple of 936 can be found by dividing 10 000 by 936. Since $\frac{10\,000}{936} \approx 10.68$, the first five-digit multiple is $936 \times 11 = 10\,296$. This means the integers from 10 296 to $10\,296 + 17 = 10\,313$ have the same remainder when divided by 18, 24, and 39.

The largest five-digit integer that is a multiple of 936 can be found by dividing 100 000 by 936. Since $\frac{100\,000}{936} \approx 106.84$, the largest five-digit multiple is $936 \times 106 = 99\,216$. This means the integers from 99 216 to $99\,216 + 17 = 99\,233$ have the same remainder when divided by 18, 24, and 39. We also note that these are all five-digit integers.

Thus, $936n$ is a positive five-digit integer for $11 \le n \le 106$. The number of positive five-digit integers that are divisible by 936 is $106 - 11 + 1 = 96$. For each of these multiples of 936, there are 18 integers that have the same remainder when divided by 18, 24, and 39. This gives a total of $96 \times 18 = 1728$ integers that have the same remainder when divided by 18, 24, and 39.

However, we need to check integers near 10 000. The largest multiple of 936 that is less than 10 000 is $936 \times 10 = 9360$. This means the integers between 9360 and $9360 + 18 = 9378$ have the same remainder when divided by 18, 24, and 39. However, none of these are five-digit integers.

Therefore, the number of five-digit positive integers that have the same remainder when divided by 18, 24, and 39 is 1728.

Problem of the Week
Problem E
Discarding Digits

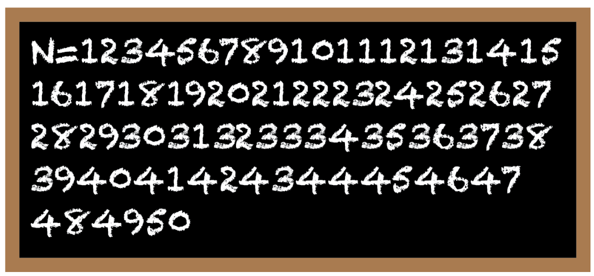
Stef forms the integer N by writing the integers from 1 to 50 in order.

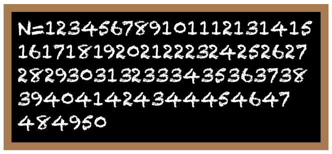
That is,

$N = 1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.$

Stef then selects some of the digits in N and discards them, so that the remaining digits, in their original order, form a new integer. The sum of the digits in this new integer is 200.

If M is the largest integer that Stef could have formed, what are the first ten digits of M?





Problem of the Week
Problem E and Solution
Discarding Digits

Problem

Stef forms the integer N by writing the integers from 1 to 50 in order.

That is,

$N = 1234567891011121314151617181920212223242526272829303132333435363738394041424344454647484950.$

Stef then selects some of the digits in N and discards them, so that the remaining digits, in their original order, form a new integer. The sum of the digits in this new integer is 200.

If M is the largest integer that Stef could have formed, what are the first ten digits of M?

Solution

We start by determining the sum of the digits of N. This is the same as determining the sum of the digits of the numbers from 1 to 50. The digits 0 to 9 sum to
$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$.

The digits in the numbers 10 to 19 sum to
$(1 + 0) + (1 + 1) + (1 + 2) + (1 + 3) + (1 + 4) + (1 + 5) + (1 + 6) + (1 + 7) + (1 + 8) + (1 + 9)$
$= 10(1) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 10 + 45 = 55$.

The digits in the numbers 20 to 29 sum to
$(2 + 0) + (2 + 1) + (2 + 2) + (2 + 3) + (2 + 4) + (2 + 5) + (2 + 6) + (2 + 7) + (2 + 8) + (2 + 9)$
$= 10(2) + (0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) = 20 + 45 = 65$.

Similarly, the digits in the numbers 30 to 39 sum to $10(3) + 45 = 75$ and the digits in the numbers 40 to 49 sum to $10(4) + 45 = 85$.

We must add 5 and 0 in order to account for the number 50 at the end of N. Therefore, the sum of the digits of N is $45 + 55 + 65 + 75 + 85 + (5 + 0) = 330$.

Since the digits of M sum to 200, the digits that are removed and discarded must sum to $330 - 200 = 130$.

In order for M to be as large as possible, we need M to have as many digits as possible. So we need to remove as few digits as possible such that the digits that are removed sum to 130. To determine the fewest number of digits to remove, we remove the largest digits in N.

We notice that in N there are five 9s, five 8s, and five 7s. These 15 digits have a sum of $5 \times 9 + 5 \times 8 + 5 \times 7 = 120$. After removing these digits, we would still need to remove additional digits that have a sum of $130 - 120 = 10$. We would need at least two more digits to do this. So the fewest number of digits that we can remove that will have a sum of 130 is 17.

So which 17 digits do we remove? We are left with the following four options.

1. Remove five 9s, five 8s, five 7s, one 6, and one 4.
2. Remove five 9s, five 8s, five 7s, and two 5s.
3. Remove five 9s, five 8s, four 7s, two 6s, and one 5.
4. Remove five 9s, four 8s, five 7s, and three 6s.

These are the only ways to remove 17 digits from N that have a sum of 130. Thus, each option will result in a number that is exactly 17 digits shorter than N. So to determine which option results in the largest possible number, we can look at how each affects the first few digits of N.

Option 1: Remove five 9s, five 8s, five 7s, one 6, and one 4.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$.

- Removing a 6 and a 4 from anywhere after the first six digits will result in a number whose first six digits are 123456.
- Removing the first 6, and a 4 from anywhere past this 6 will result in a number whose first 6 digits are 123451.
- Removing the first 4, and a 6 from anywhere else in the number will result in a number whose first 6 digits are 123510 or 123561.

Since $123561 > 123510 > 123456 > 123451$, removing the first 4, and a 6 from anywhere else in the number can result in a number whose first six digits are 123561. This would result in the largest possible number so far.

Option 2: Remove five 9s, five 8s, five 7s, and two 5s.
After removing all the 9s, 8s, and 7s, the remaining digits start $123456101112\ldots$. No matter how we remove two 5s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 3: Remove five 9s, five 8s, four 7s, two 6s, and one 5.
After removing all the 9s and 8s, the remaining digits start $1234567101112\ldots$. No matter how we remove four 7s, two 6s, and one 5, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Option 4: Remove five 9s, four 8s, five 7s, and three 6s.
After removing all the 9s and 7s, the remaining digits start $1234568101112\ldots$. No matter how we remove four 8s, and three 6s, we are not able to get a number whose first six digits are larger than 123561. Thus, this option will not result in the largest possible value of M.

Therefore, in order to form the largest possible value of M, we should remove all 9s, 8s, and 7s, the first 4, and a 6 from anywhere else in the number.

After removing the 9s, 8s, 7s, and the first 4, we are left with

$$12356101112131415161112021222324252622230313233343536333404142434445464444 50$$

We still must remove a 6 to bring our digit sum to 200. We want whatever 6 we remove to affect the size of the final number in the least possible way. We need to therefore remove the 6 with the lowest place value. The 6 to be removed is therefore the hundred thousands digit in the number shown just above. After removing this 6,

$$M = 1235610111213141516111202122232425262223031323334353633340414243444544 4450$$

It follows that the first 10 digits of M are 1235610111.

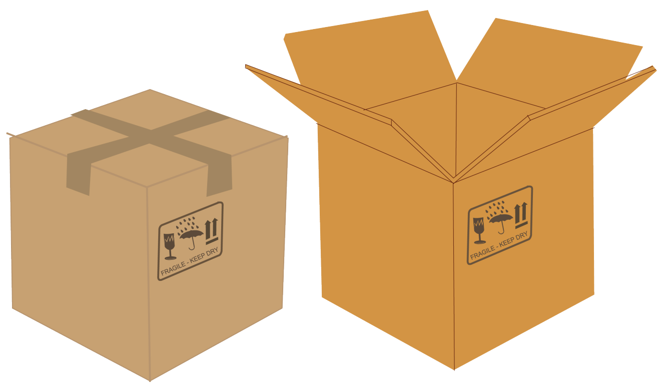
Problem of the Week
Problem E
Picking Boxes

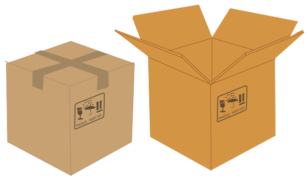
Billy's Box Company sells boxes with the following very particular restrictions on their dimensions.

- The length, width, and height, in cm, must be all integers.
- The ratio of the length to the width to the height must be $4 : 3 : 5$.
- The sum of the length, width, and height must be between 100 cm and 1000 cm, inclusive.

Stefan bought the box with the smallest possible volume, and Lali bought the box with the largest volume less than 4 m^3.

Determine the dimensions of Stefan and Lali's boxes.





Problem of the Week
Problem E and Solution
Picking Boxes

Problem

Billy's Box Company sells boxes with the following very particular restrictions on their dimensions.

- The length, width, and height, in cm, must be all integers.
- The ratio of the length to the width to the height must be $4 : 3 : 5$.
- The sum of the length, width, and height must be between 100 cm and 1000 cm, inclusive.

Stefan bought the box with the smallest possible volume, and Lali bought the box with the largest volume less than 4 m^3.

Determine the dimensions of Stefan and Lali's boxes.

Solution

Since the boxes from Billy's Box Company have integer side lengths in the ratio $4 : 3 : 5$, let $4n$ represent the length of a box in cm, let $3n$ represent the width of a box in cm, and let $5n$ represent the height of a box in cm, where n is an integer.

Furthermore, the sum of the length, width and height must be at least 100 cm. It follows that

$$\begin{aligned} 4n + 3n + 5n &\geq 100 \\ 12n &\geq 100 \\ n &\geq \frac{100}{12} = 8\frac{1}{3} \end{aligned}$$

Also, the sum of the length, width and height must be at most 1000 cm. It follows that

$$\begin{aligned} 4n + 3n + 5n &\leq 1000 \\ 12n &\leq 1000 \\ n &\leq \frac{1000}{12} = 83\frac{1}{3} \end{aligned}$$

There is one other restriction to consider, since the volume of Lali's box is less than 4 m^3. To convert from m^3 to cm^3, note that

$$\begin{aligned} 1\text{ m}^3 &= 1\text{ m} \times 1\text{ m} \times 1\text{ m} \\ &= 100\text{ cm} \times 100\text{ cm} \times 100\text{ cm} \\ &= 1\,000\,000\text{ cm}^3 \end{aligned}$$

Therefore, $4\text{ m}^3 = 4\,000\,000\text{ cm}^3$.

It follows that

$$\begin{aligned} (4n)(3n)(5n) &< 4\,000\,000 \\ 60n^3 &< 4\,000\,000 \\ n^3 &< \frac{200\,000}{3} \\ n &< \sqrt[3]{\frac{200\,000}{3}} \approx 40.5 \end{aligned}$$

We also know that n is an integer. Since $n \geq 8\frac{1}{3}$, then the smallest possible integer value of n is 9. Using the dimensions $4n$, $3n$, and $5n$ with $n = 9$, we can determine that the dimensions of Stefan's box are 36 cm by 27 cm by 45 cm.

For Lali's box, since $n \leq 83\frac{1}{3}$ and $n < 40.5$, then the largest possible value of n is 40. Using the dimensions $4n$, $3n$, and $5n$ with $n = 40$, we can determine that the dimensions of Lali's box are 160 cm by 120 cm by 200 cm. This box has a volume of 3.84 m^3.

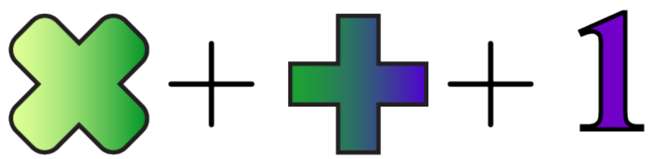
Problem of the Week

Problem E

Bilal's Choices

Bilal chooses two distinct positive integers. He adds the product of the integers to the sum of the integers, and then adds 1. He finds that the result is equal to 196.

Determine all possible pairs of integers that Bilal could have chosen.



Problem of the Week
Problem E and Solution
Bilal's Choices

Problem

Bilal chooses two distinct positive integers. He adds the product of the integers to the sum of the integers, and then adds 1. He finds that the result is equal to 196.

Determine all possible pairs of integers that Bilal could have chosen.

Solution

Let x and y represent the two positive integers that Bilal chooses. Since the integers are distinct, $x \neq y$. Let $x < y$. That is, let x represent the smaller of the two integers.
The product of the two integers is xy and the sum is $(x + y)$.
Bilal adds the product of the numbers to the sum of the numbers and then adds 1, and the result is 196. Thus,

$$xy + x + y + 1 = 196$$

Factoring the left side, by grouping the first two terms and the last two terms, we get

$$\begin{aligned} x(y+1) + 1(y+1) &= 196 \\ (x+1)(y+1) &= 196 \end{aligned}$$

Since x and y are positive integers, then $x + 1$ and $y + 1$ are positive integers. Thus, we are looking for a pair of positive integers whose product is 196. There are four ways to factor 196 as a product of two positive integers:

$$196 = 1 \times 196 = 2 \times 98 = 4 \times 49 = 7 \times 28 = 14 \times 14$$

For the product $196 = 1 \times 196$, we have $x + 1 = 1$ and $y + 1 = 196$. Thus, $x = 0$ and $y = 195$. Since the required numbers are positive integers, this solution is inadmissible.

For the product $196 = 2 \times 98$, we have $x + 1 = 2$ and $y + 1 = 98$. Thus, $x = 1$ and $y = 97$. This is a valid solution.

For the product $196 = 4 \times 49$, we have $x + 1 = 4$ and $y + 1 = 49$. Thus, $x = 3$ and $y = 48$. This is a valid solution.

For the product $196 = 7 \times 28$, we have $x + 1 = 7$ and $y + 1 = 28$. Thus, $x = 6$ and $y = 27$. This is a valid solution.

For the product $196 = 14 \times 14$, we have $x + 1 = 14$ and $y + 1 = 14$. Thus, $x = 13$ and $y = 13$. Since the required numbers are distinct, this solution is inadmissible.

Therefore, there are three pairs of distinct positive integers that Bilal could have chosen: 1 and 97, 3 and 48, or 6 and 27. It can be shown that these three pairs do indeed each give the required result.

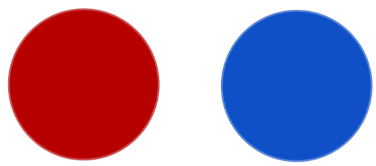
Problem of the Week

Problem E

Red and Blue Chips

Jane and Fred each have their own collection of red and blue bingo chips. The ratio of the number of Jane's chips to the number of Fred's chips is $3 : 2$. When they combine their chips, the ratio of the number of red chips to the number of blue chips is $7 : 3$. For Jane's chips, the ratio of the number of red chips to the number of blue chips is $4 : 1$.

What is the ratio of the number of red chips to the number of blue chips for Fred's bingo chips?



Problem of the Week

Problem E and Solution

Red and Blue Chips

Problem

Jane and Fred each have their own collection of red and blue bingo chips. The ratio of the number of Jane's chips to the number of Fred's chips is $3 : 2$. When they combine their chips, the ratio of the number of red chips to the number of blue chips is $7 : 3$. For Jane's chips, the ratio of the number of red chips to the number of blue chips is $4 : 1$.

What is the ratio of the number of red chips to the number of blue chips for Fred's bingo chips?

Solution

Solution 1

Suppose that Jane and Fred have a total of 100 bingo chips. (We may assume any convenient total number of chips.)

Since the ratio of the number of Jane's chips to the number of Fred's chips is $3 : 2$, then Jane has $\frac{3}{5}$ of the 100 chips, or 60 chips. Fred has the remaining 40 chips.

When they combine their chips, the ratio of the number of red chips to the number of blue chips of $7 : 3$. Therefore, $\frac{7}{10}$ of the 100 chips, or 70 chips, are red and the remaining 30 chips are blue.

For Jane's chips, the ratio of the number of red chips to the number of blue chips is $4 : 1$, so $\frac{4}{5}$ of her 60 chips, or 48 chips, are red and the remaining 12 chips are blue.

Since there are 70 red chips in total, then Fred has $70 - 48 = 22$ red chips.
Since there are 30 blue chips in total, then Fred has $30 - 12 = 18$ blue chips.

Therefore, the ratio of the number of red chips to the number of blue chips for Fred's chips is $22 : 18 = 11 : 9$.

Solution 2

Suppose that Jane and Fred have a total of x chips.

Since the ratio of the number of Jane's chips to the number of Fred's chips is $3 : 2$, then Jane has $\frac{3}{5}$ of the chips, or $\frac{3}{5}x$ chips. Fred has the remaining $\frac{2}{5}x$ chips.

When they combine their chips, the ratio of the number of red chips to the number of blue chips is $7 : 3$. Therefore, $\frac{7}{10}x$ chips are red and the remaining $\frac{3}{10}x$ chips are blue.

For Jane's chips, the ratio of the number of red chips to the number of blue chips is $4 : 1$, so $\frac{4}{5}$ of her $\frac{3}{5}x$ chips, or $\frac{4}{5}\left(\frac{3}{5}x\right) = \frac{12}{25}x$, are red and the remaining $\frac{3}{5}x - \frac{12}{25}x = \frac{3}{25}x$ chips are blue.

Since there are $\frac{7}{10}x$ red chips in total, then Fred has $\frac{7}{10}x - \frac{12}{25}x = \frac{11}{50}x$ red chips.

Since there are $\frac{3}{10}x$ blue chips in total, then Fred has $\frac{3}{10}x - \frac{3}{25}x = \frac{9}{50}x$ blue chips.

Therefore, the ratio of the number of red chips to the number of blue chips for Fred's chips is $\frac{11}{50}x : \frac{9}{50}x = 11 : 9$.

Problem of the Week

Problem E

Coin Combinations

In Canada, a \$2 coin is called a toonie, a \$1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of \$1.

How many different combinations of toonies, loonies, and/or quarters have a total value of \$100?

NOTE: In solving this problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

For example, the sum of the first 10 positive integers is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = \frac{10(10+1)}{2} = \frac{10(11)}{2} = 55$$

Problem of the Week
Problem E and Solution
Coin Combinations

Problem

In Canada, a \$2 coin is called a toonie, a \$1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of \$1.

How many different combinations of toonies, loonies, and/or quarters have a total value of \$100?

Solution

We will break the solution into cases based on the number of \$2 coins used. For each case, we will count the number of possibilities for the number of \$1 and 25¢ coins.

The maximum number of \$2 coins we can use is 50, since $\$2 \times 50 = \$100$. If we use 50 \$2 coins, then we do not need any \$1 or 25¢ coins. Therefore, there is only one way to make a total of \$100 if there are 50 \$2 coins.

Suppose we use 49 \$2 coins. Since $\$2 \times 49 = \$98$, to reach a total of \$100, we would need two \$1 and no 25¢ coins, or one \$1 and four 25¢ coins, or no \$1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of \$100 if we use 49 \$2 coins.

Suppose we use 48 \$2 coins. Since $\$2 \times 48 = \$96$, to reach a total of \$100, we would need four \$1 and no 25¢ coins, or three \$1 and four 25¢ coins, or two \$1 and eight 25¢ coins, or one \$1 and twelve 25¢ coins, or no \$1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of \$100 if we use 48 \$2 coins.

We start to see a pattern. When we reduce the number of \$2 coins by one, the number of possible combinations using that many \$2 coins increases by 2. This is because there are 2 more options for the number of \$1 coins we can use. Thus, when we use 47 \$2 coins, there are 7 possible ways to make a total of \$100. When we use 46 \$2 coins, there are 9 possible ways to make a total of \$100, and so on. When we use 1 \$2 coin, there are 99 different ways to make the difference of \$98 (because you can use 0 to 98 \$1 coins). When we don't use any \$2 coins, there are 101 different ways to make a total of \$100 (because you can use 0 to 100 \$1 coins). Thus, the number of different combinations of coins that have a total value of \$100 is

$$1 + 3 + 5 + 7 + 9 + \cdots + 99 + 101$$

Adding and subtracting the even numbers from 2 to 100, we get

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \cdots + 98 + 99 + 100 + 101 - (2 + 4 + 6 + 8 + \cdots + 98 + 100)$$

Factoring out a factor of 2 from the subtracted even numbers, we get

$$(1 + 2 + 3 + \cdots + 100 + 101) - 2(1 + 2 + 3 + 4 + \cdots + 50)$$

We can then use the formula for the sum of the first n positive integers to find that this expression is equal to

$$\frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) = 101(51) - 50(51) = 2601$$

Therefore, there are 2601 different combinations of toonies, loonies, and/or quarters that have a total value of \$100.

Extension:

Let's look at the end of the previous computation another way.

$$\begin{aligned} 1+3+5+7+9+\cdots+99+101 &= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) \\ &= 101(51) - 50(51) \\ &= 51(101-50) \\ &= 51(51) \\ &= 51^2 \end{aligned}$$

How many odd integers are in the list from 1 to 101? From 1 to 101, there are 101 integers. This list contains the even integers, from 2 to 100, which are 50 in total. Therefore, there are $101 - 50 = 51$ odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is equal to 51^2? Is the sum of the first 1000 odd positive integers equal to 1000^2? Is the sum of the first n odd positive integers equal to n^2?

We will develop a formula for the sum of the first n odd positive integers.

We saw in the problem statement that the sum of the first n positive integers is

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Every odd positive integer can be written in the form $2n-1$, where n is an integer ≥ 1. When $n=1,\ 2n-1 = 2(1)-1 = 1$; when $n=2,\ 2n-1 = 2(2)-1 = 3$, and so on. So the 51^{st} odd positive integer is $2(51)-1 = 101$, as we determined above. The n^{th} odd positive integer is $2n-1$. Let's consider the sum of the first n odd positive integers. That is,

$$1+3+5+7+\cdots+(2n-3)+(2n-1)$$

Adding and subtracting the even numbers from 2 to $2n$, we get

$$\begin{aligned} &1+2+3+4+5+\cdots+(2n-3)+(2n-2)+(2n-1)+2n-(2+4+6+\cdots+(2n-2)+2n) \\ &= (1+2+3+4+\cdots+2n) - (2+4+6+8+\cdots+(2n-2)+2n) \end{aligned}$$

Factoring out a 2 from the subtracted even numbers, we get

$$(1+2+3+4+\cdots+2n) - 2(1+2+3+\cdots+n)$$

We can then use the formula for the sum of the first n positive integers to find that this expression is equal to

$$\begin{aligned} \frac{2n(2n+1)}{2} - 2\left(\frac{n(n+1)}{2}\right) &= n(2n+1) - n(n+1) \\ &= 2n^2 + n - n^2 - n \\ &= n^2 \end{aligned}$$

Therefore, the sum of the first n odd positive integers is equal to n^2.

For Further Thought: Can you develop a formula for the sum of the first n even positive integers?

Problem of the Week

Problem E

It's the Ones that We Want

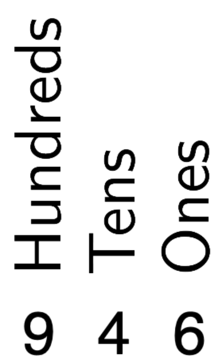
The sum of the first n positive integers is $1 + 2 + 3 + \cdots + n$.
We define a_n to be the ones digit of the sum of the first n positive integers.

For example,

$$\begin{aligned} 1 &= 1 \quad \text{and} \quad a_1 = 1, \\ 1 + 2 &= 3 \quad \text{and} \quad a_2 = 3, \\ 1 + 2 + 3 &= 6 \quad \text{and} \quad a_3 = 6, \\ 1 + 2 + 3 + 4 &= 10 \quad \text{and} \quad a_4 = 0, \\ 1 + 2 + 3 + 4 + 5 &= 15 \quad \text{and} \quad a_5 = 5. \end{aligned}$$

Thus, $a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 6 + 0 + 5 = 15$.

Determine the smallest value of n such that $a_1 + a_2 + a_3 + \cdots + a_n \geq 2024$.



Hundreds Tens Ones
9 4 6

Problem of the Week
Problem E and Solution
It's the Ones that We Want

Problem

The sum of the first n positive integers is $1 + 2 + 3 + \cdots + n$.
We define a_n to be the ones digit of the sum of the first n positive integers.

For example,

$$\begin{aligned} 1 &= 1 \text{ and } a_1 = 1, \\ 1 + 2 &= 3 \text{ and } a_2 = 3, \\ 1 + 2 + 3 &= 6 \text{ and } a_3 = 6, \\ 1 + 2 + 3 + 4 &= 10 \text{ and } a_4 = 0, \\ 1 + 2 + 3 + 4 + 5 &= 15 \text{ and } a_5 = 5. \end{aligned}$$

Thus, $a_1 + a_2 + a_3 + a_4 + a_5 = 1 + 3 + 6 + 0 + 5 = 15$.

Determine the smallest value of n such that $a_1 + a_2 + a_3 + \cdots + a_n \geq 2024$.

Solution

Let's start by examining the values of a_n until we start to see a pattern.

We know $a_1 = 1$, $a_2 = 3$, $a_3 = 6$, $a_4 = 0$, and $a_5 = 5$.

Unfortunately, we do not have a pattern yet. We need to keep calculating values of a_n. Since $15 + 6 = 21$, $a_6 = 1$.

Notice that we can determine the ones digit of the sum of the first n integers from the ones digit from the sum of the first $n - 1$ integers and the ones digit of n. For example, to calculate a_7, we simply need to know that $a_6 = 1$ and the sum $1 + 7 = 8$ has ones digit 8. So $a_7 = 8$. Thus, continuing on, we know

$$\begin{aligned} a_8 &= 6, \text{ since } a_7 + 8 = 16 \\ a_9 &= 5, \text{ since } a_8 + 9 = 15 \\ a_{10} &= 5, \text{ since } a_9 + 0 = 5 \\ a_{11} &= 6, \text{ since } a_{10} + 1 = 6 \\ a_{12} &= 8, \text{ since } a_{11} + 2 = 8 \\ a_{13} &= 1, \text{ since } a_{12} + 3 = 11 \\ a_{14} &= 5, \text{ since } a_{13} + 4 = 5 \\ a_{15} &= 0, \text{ since } a_{14} + 5 = 10 \\ a_{16} &= 6, \text{ since } a_{15} + 6 = 6 \\ a_{17} &= 3, \text{ since } a_{16} + 7 = 13 \\ a_{18} &= 1, \text{ since } a_{17} + 8 = 11 \\ a_{19} &= 0, \text{ since } a_{18} + 9 = 10 \\ a_{20} &= 0, \text{ since } a_{19} + 0 = 0 \\ a_{21} &= 1, \text{ since } a_{20} + 1 = 1 \end{aligned}$$

The values of a_n should repeat now. Can you see why?
Since $a_{21} = a_1$ and the ones digit of 22 equals the ones digit of 2, $a_{22} = a_2$.
Similarly, since $a_{22} = a_2$ and the ones digit of 23 equals the ones digit of 3, $a_{23} = a_3$.
We will also have $a_{24} = a_4$, and so on.

Therefore, the values of a_n will repeat every 20 values of n.

We can calculate

$$\begin{aligned}&a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10} + a_{11} + a_{12} + a_{13} + a_{14} + a_{15} + a_{16} + a_{17} + a_{18} + a_{19} + a_{20}\\ &\quad = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5 + 6 + 8 + 1 + 5 + 0 + 6 + 3 + 1 + 0 + 0\\ &\quad = 70\end{aligned}$$

Since the values of a_n repeat every 20 values of n, it is also true that
$a_{21} + a_{22} + a_{23} + \cdots + a_{39} + a_{40} = 70$, and $a_{41} + a_{42} + a_{43} + \cdots + a_{59} + a_{60} = 70$, and so on.
Since $\frac{2024}{70} = 28\frac{32}{35}$, there are 28 complete cycles of the 20 repeating values of a_n.
Therefore, the sum of the first $28 \times 20 = 560$ values of a_n sum to $28 \times 70 = 1960$.

In other words, $a_1 + a_2 + a_3 + \cdots + a_{559} + a_{560} = 1960$.

Let's keep adding values of a_n until we reach 2024.

$$\begin{aligned}a_{561} + a_{562} + a_{563} + a_{564} + a_{565} + a_{566} + a_{567} + a_{568} + a_{569} + a_{570}\\ = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + a_{10}\\ = 1 + 3 + 6 + 0 + 5 + 1 + 8 + 6 + 5 + 5\\ = 40\end{aligned}$$

Therefore, $a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} = 1960 + 40 = 2000$.

We also know that $a_{571} = a_{11} = 6$, $a_{572} = a_{12} = 8$, $a_{573} = a_{13} = 1$, $a_{574} = a_{14} = 5$, and $a_{575} = a_{15} = 0$.

Thus,

$$\begin{aligned}a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} + a_{575} &= 2000 + 6 + 8 + 1 + 5 + 0\\ &= 2020 \le 2024\end{aligned}$$

and

$$\begin{aligned}a_1 + a_2 + a_3 + \cdots + a_{569} + a_{570} + a_{571} + a_{572} + a_{573} + a_{574} + a_{575} + a_{576} &= 2020 + 6\\ &= 2026 \ge 2024\end{aligned}$$

Therefore, the smallest value of n such that $a_1 + a_2 + a_3 + \cdots + a_n \ge 2024$ is $n = 576$.

Problem of the Week

Problem E

The Weightiest Problem

Pauline has five rocks, each of a different mass. She weighs the rocks in pairs and records the mass of each pair of rocks. Later, she realizes that she forgot to weigh the rocks individually, but no longer has access to a scale. The ten recorded masses are 728 g, 757 g, 771 g, 783 g, 797 g, 817 g, 826 g, 831 g, 860 g, and 886 g. Determine the combined mass of all five rocks. Then determine the mass of the lightest rock.

Problem of the Week
Problem E and Solution
The Weightiest Problem

Problem

Pauline has five rocks, each of a different mass. She weighs the rocks in pairs and records the mass of each pair of rocks. Later, she realizes that she forgot to weigh the rocks individually, but no longer has access to a scale. The ten recorded masses are 728 g, 757 g, 771 g, 783 g, 797 g, 817 g, 826 g, 831 g, 860 g, and 886 g. Determine the combined mass of all five rocks. Then determine the mass of the lightest rock.

Solution

Let a, b, c, d, and e represent the masses of the rocks, in grams, from lightest to the heaviest.

Notice that
$(a + b) + (a + c) + (a + d) + (a + e) + (b + c) + (b + d) + (b + e) + (c + d) + (c + e) + (d + e)$ simplifies to $4a + 4b + 4c + 4d + 4e$. It is also equal to the sum of the 10 recorded masses.

Thus,

$$\begin{aligned} 4a + 4b + 4c + 4d + 4e &= 728 + 757 + 771 + 783 + 797 + 817 + 826 + 831 + 860 + 886 \\ &= 8056 \end{aligned}$$

Therefore, $4a + 4b + 4c + 4d + 4e = 8056$. Dividing each side of the equation by 4, we obtain $a + b + c + d + e = 2014$. That is, the combined mass of all five rocks is 2014 g.

The smallest recorded mass is created by weighing the two lightest rocks. Therefore, $a + b = 728$. The largest recorded mass is created by weighing the two heaviest rocks. Therefore, $d + e = 886$. Thus,

$$\begin{aligned} (a + b + c + d + e) - (a + b) - (d + e) &= 2014 - 728 - 886 \\ c &= 400 \end{aligned}$$

The second smallest recorded mass is created by adding the mass of the lightest object, a, to the mass of the object with the mass in the middle, c. (A justification of this statement is given at the end of this solution.) Therefore, $a + c = 757$ and $a = 757 - c = 757 - 400 = 357$.

Therefore, the combined mass of all five rocks is 2014 g and the lightest rock has mass 357 g.

Although we are not asked to, from here, we could go on to determine the masses of all five rocks. Doing so, we would find that the masses are 357 g, 371 g, 400 g, 426 g, and 460 g. It is left as an exercise for you to verify the correctness of this list.

Why is $a + c$ the second smallest recorded mass?

We will repeatedly use the fact that $a < b < c < d < e$ to show that $a + c$ is the second smallest recorded mass. First, since $c > b$, $a + c > a + b$. That is, $a + c$ is not the smallest recorded mass. We'll now show that $a + c$ is smaller than all other pairs.

- Since $c < d$, we have $a + c < a + d$.
- Since $c < e$, we have $a + c < a + e$.
- Since $a < b$, we have $a + c < b + c$.
- Since $a < b$ and $c < d$, we have $a + c < b + d$.
- Since $a < b$ and $c < e$, we have $a + c < b + e$.
- Since $a < d$, we have $a + c < d + c = c + d$.
- Since $a < e$, we have $a + c < e + c = c + e$.
- Since $a < d$ and $c < e$, we have $a + c < d + e$.

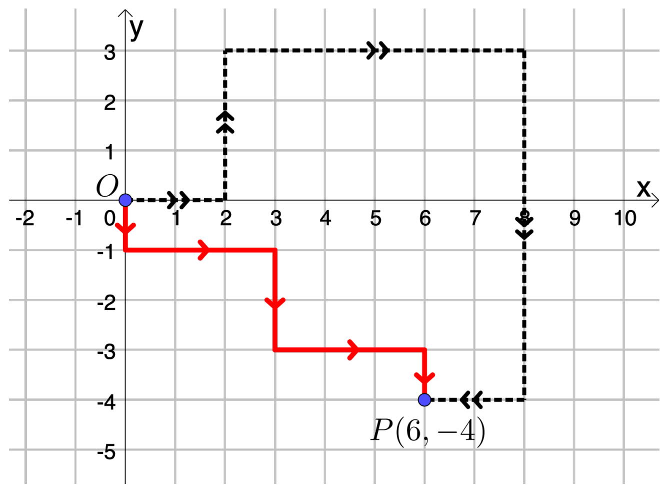
We have shown that $a + c > a + b$ and that $a + c$ is smaller than the other eight recorded sums. Therefore, $a + c$ is the second smallest recorded mass. In a similar manner, it is also possible to show that $c + e$ is the second largest recorded mass.

Problem of the Week

Problem E

The Shortest Path

On the Cartesian plane, we draw grid lines at integer points along the x and y axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0,0)$ to $P(6,-4)$. One path has length 10 and the other has length 20.



There are many different paths along the grid lines from O to P, but the smallest possible length of such a path is 10. Let's call this smallest possible length the *path distance* from O to P.

Determine the number of points with integer coordinates for which the path distance from O to that point is 10.

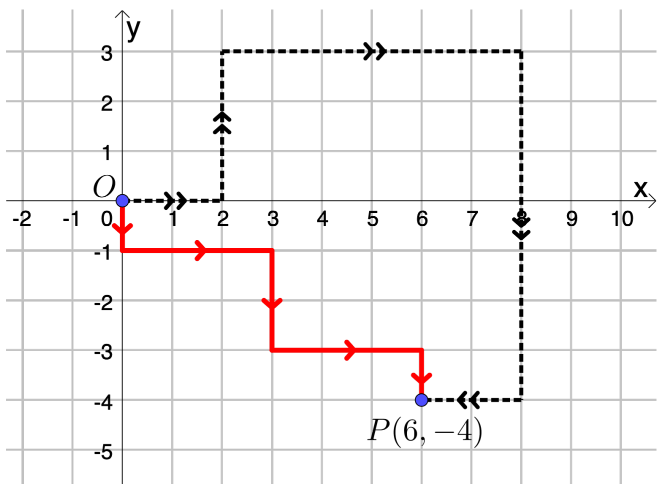
Problem of the Week

Problem E and Solution

The Shortest Path

Problem

On the Cartesian plane, we draw grid lines at integer points along the x and y axes. We can then draw paths along these grid lines between any two points with integer coordinates. The graph below shows two paths along these grid lines from $O(0, 0)$ to $P(6, -4)$. One path has length 10 and the other has length 20.



There are many different paths along the grid lines from O to P, but the smallest possible length of such a path is 10. Let's call this smallest possible length the *path distance* from O to P.

Determine the number of points with integer coordinates for which the path distance from O to that point is 10.

Solution

Solution 1

Let $Q(a, b)$ be a point that has path distance 10 from $O(0, 0)$.

Let's first suppose that Q lies on the x or y axis.
The only point along the positive x-axis that has path distance 10 from the origin is $(10, 0)$.
The only point along the negative x-axis that has path distance 10 from the origin is $(-10, 0)$.
The only point along the positive y-axis that has path distance 10 from the origin is $(0, 10)$.
The only point along the negative y-axis that has path distance 10 from the origin is $(0, -10)$.
Therefore, there are 4 points along the axes that have a path distance 10 from O.

Next, let's suppose $a > 0$ and $b > 0$, so Q is in the first quadrant.
Since the path distance from O to Q is 10, there must be a path from O to Q that moves a total of r units to the right and u units up (in some order) such that $r + u = 10$. This means that Q is r units to the right of O and u units up from O. In other words, $a = r$ and $b = u$, so $a + b = r + u = 10$.

The points (a, b) in the first quadrant that satisfy $a + b = 10$ where a and b are integers are $(1, 9)$, $(2, 8)$, $(3, 7)$, $(4, 6)$, $(5, 5)$, $(6, 4)$, $(7, 3)$, $(8, 2)$, $(9, 1)$. There are 9 such pairs. Therefore, there are 9 points in the first quadrant that have path distance 10 from O.

By symmetry, there are 9 points in each quadrant that have path distance 10 from O.
In quadrant 2, the points are $(-1, 9)$, $(-2, 8)$, $(-3, 7)$, $(-4, 6)$, $(-5, 5)$, $(-6, 4)$, $(-7, 3)$, $(-8, 2)$, $(-9, 1)$. In quadrant 3, the points are $(-1, -9)$, $(-2, -8)$, $(-3, -7)$, $(-4, -6)$, $(-5, -5)$, $(-6, -4)$, $(-7, -3)$, $(-8, -2)$, $(-9, -1)$. In quadrant 4, the points are $(1, -9)$, $(2, -8)$, $(3, -7)$, $(4, -6)$, $(5, -5)$, $(6, -4)$, $(7, -3)$, $(8, -2)$, $(9, -1)$.

Therefore, there are a total of $4 + (4 \times 9) = 40$ points with integer coordinates that have path distance 10 from O.

Solution 2

We are permitted 10 moves to get from the origin to a point by travelling along the grid lines. These moves can be all horizontal (in one direction), all vertical (in one direction), or a combination of horizontal moves (in one direction) with vertical moves (in one direction).

We examine the cases based on the number of horizontal moves.

- **0 horizontal moves**: Since there are 0 horizontal moves, there are 10 vertical moves. There are two possible endpoints, $(0, 10)$ and $(0, -10)$.
- **1 horizontal move**: Since there is 1 horizontal move, there are 9 vertical moves. There are four possible endpoints, $(-1, 9)$, $(-1, -9)$, $(1, 9)$, and $(1, -9)$.
- **2 horizontal moves**: Since there are 2 horizontal moves, there are 8 vertical moves. There are four possible endpoints, $(-2, 8)$, $(-2, -8)$, $(2, 8)$, and $(2, -8)$.
- **3 horizontal moves**: Since there are 3 horizontal moves, there are 7 vertical moves. There are four possible endpoints, $(-3, 7)$, $(-3, -7)$, $(3, 7)$, and $(3, -7)$.
- **4 horizontal moves**: Since there are 4 horizontal moves, there are 6 vertical moves. There are four possible endpoints, $(-4, 6)$, $(-4, -6)$, $(4, 6)$, and $(4, -6)$.
- **5 horizontal moves**: Since there are 5 horizontal moves, there are 5 vertical moves. There are four possible endpoints, $(-5, 5)$, $(-5, -5)$, $(5, 5)$, and $(5, -5)$.
- **6 horizontal moves**: Since there are 6 horizontal moves, there are 4 vertical moves. There are four possible endpoints, $(-6, 4)$, $(-6, -4)$, $(6, 4)$, and $(6, -4)$.
- **7 horizontal moves**: Since there are 7 horizontal moves, there are 3 vertical moves. There are four possible endpoints, $(-7, 3)$, $(-7, -3)$, $(7, 3)$, and $(7, -3)$.
- **8 horizontal moves**: Since there are 8 horizontal moves, there are 2 vertical moves. There are four possible endpoints, $(-8, 2)$, $(-8, -2)$, $(8, 2)$, and $(8, -2)$.
- **9 horizontal moves**: Since there are 9 horizontal moves, there is 1 vertical move. There are four possible endpoints, $(-9, 1)$, $(-9, -1)$, $(9, 1)$, and $(9, -1)$.
- **10 horizontal moves**: Since there are 10 horizontal moves, there are 0 vertical moves. There are two possible endpoints, $(-10, 0)$ and $(10, 0)$.

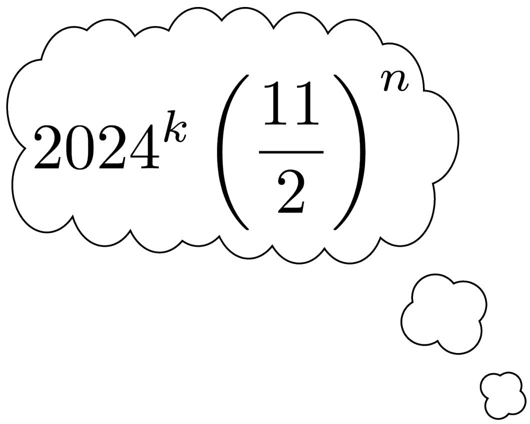
Therefore, there are a total of $2 + (4 \times 9) + 2 = 40$ points with integer coordinates that have path distance 10 from O.

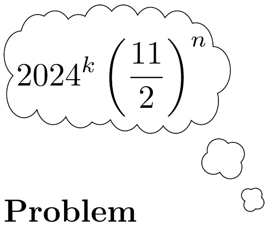
Problem of the Week

Problem E

Looking for Integers

Suppose n and k are integers and $4^k < 2024$. For how many (n, k) pairs is $2024^k \left(\frac{11}{2}\right)^n$ equal to an integer?





Problem of the Week

Problem E and Solution

Looking for Integers

Problem

Suppose n and k are integers and $4^k < 2024$. For how many (n, k) pairs is $2024^k\left(\frac{11}{2}\right)^n$ equal to an integer?

Solution

First we write 2024 as a product of prime factors: $2024 = 2^3 \times 11 \times 23$.

We can then substitute this into our expression.

$$\begin{aligned}
2024^k\left(\frac{11}{2}\right)^n &= \left(2^3 \times 11 \times 23\right)^k\left(\frac{11}{2}\right)^n \\
&= 2^{3k} \times 11^k \times 23^k \times \frac{11^n}{2^n} \\
&= 2^{3k-n} \times 11^{k+n} \times 23^k
\end{aligned}$$

Since $2024^k\left(\frac{11}{2}\right)^n$ is equal to an integer, it follows that none of the exponents can be negative. Thus, $3k - n \geq 0$, $k + n \geq 0$, and $k \geq 0$.

From $3k - n \geq 0$, we can determine that $n \leq 3k$. Similarly, from $k + n \geq 0$, we can determine that $n \geq -k$. Thus, n is an integer between $-k$ and $3k$, inclusive.

Since $4^5 = 1024$, $4^6 = 4096$, and $4^k < 2024$, it follows that $k \leq 5$. Since $k \geq 0$ and k is an integer, the possible values of k are $0,\ 1,\ 2,\ 3,\ 4$, and 5.

In the table below, we summarize the number of values of n for each possible value of k.

k	Minimum value of n	Maximum value of n	Number of values of n
0	0	0	1
1	-1	3	5
2	-2	6	9
3	-3	9	13
4	-4	12	17
5	-5	15	21

Thus, the total number of (n, k) pairs is $1 + 5 + 9 + 13 + 17 + 21 = 66$.

Problem of the Week
Problem E
Pineapples and Bananas

In a recent survey, Grade 12 students were asked if they like pineapples. They were then asked if they like bananas. It was found that

- 30% of the students do not like pineapples,
- 36 students do not like bananas,
- 60 students like both fruits, and
- 48 students like one fruit but not the other.

How many students do not like pineapples and do not like bananas?

Problem of the Week

Problem E and Solution

Pineapples and Bananas

Problem

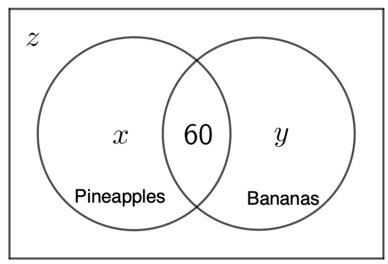
In a recent survey, Grade 12 students were asked if they like pineapples. They were then asked if they like bananas. It was found that

- 30% of the students do not like pineapples,
- 36 students do not like bananas,
- 60 students like both fruits, and
- 48 students like one fruit but not the other.

How many students do not like pineapples and do not like bananas?

Solution

Let x be the number of students that like only pineapples.
Let y be the number of students that like only bananas.
Let z be the number of students that do not like pineapples and do not like bananas.
We're given that 60 students like both pineapples and bananas.
This information is summarized in the Venn diagram.



The number of students that do not like bananas is equal to $x + z$. Therefore,

$$x + z = 36 \qquad (1)$$

The number of students who like one fruit but not the other is equal to $x + y$. Therefore,

$$x + y = 48 \qquad (2)$$

The total number of students is equal to $x + y + z + 60$ and 30% of this is $0.3(x + y + z + 60)$. This is also equal to the number of students that do not like pineapples, $y + z$. Therefore,

$$0.3(x + y + z + 60) = y + z \qquad (3)$$

Subtracting equation (1) from equation (2), we get $y - z = 12$. Therefore, $y = z + 12$.

Substituting equation (1) into equation (3) we get $0.3(36 + y + 60) = y + z$, or $0.3(96 + y) = y + z$.

Now substituting $y = z + 12$, we get

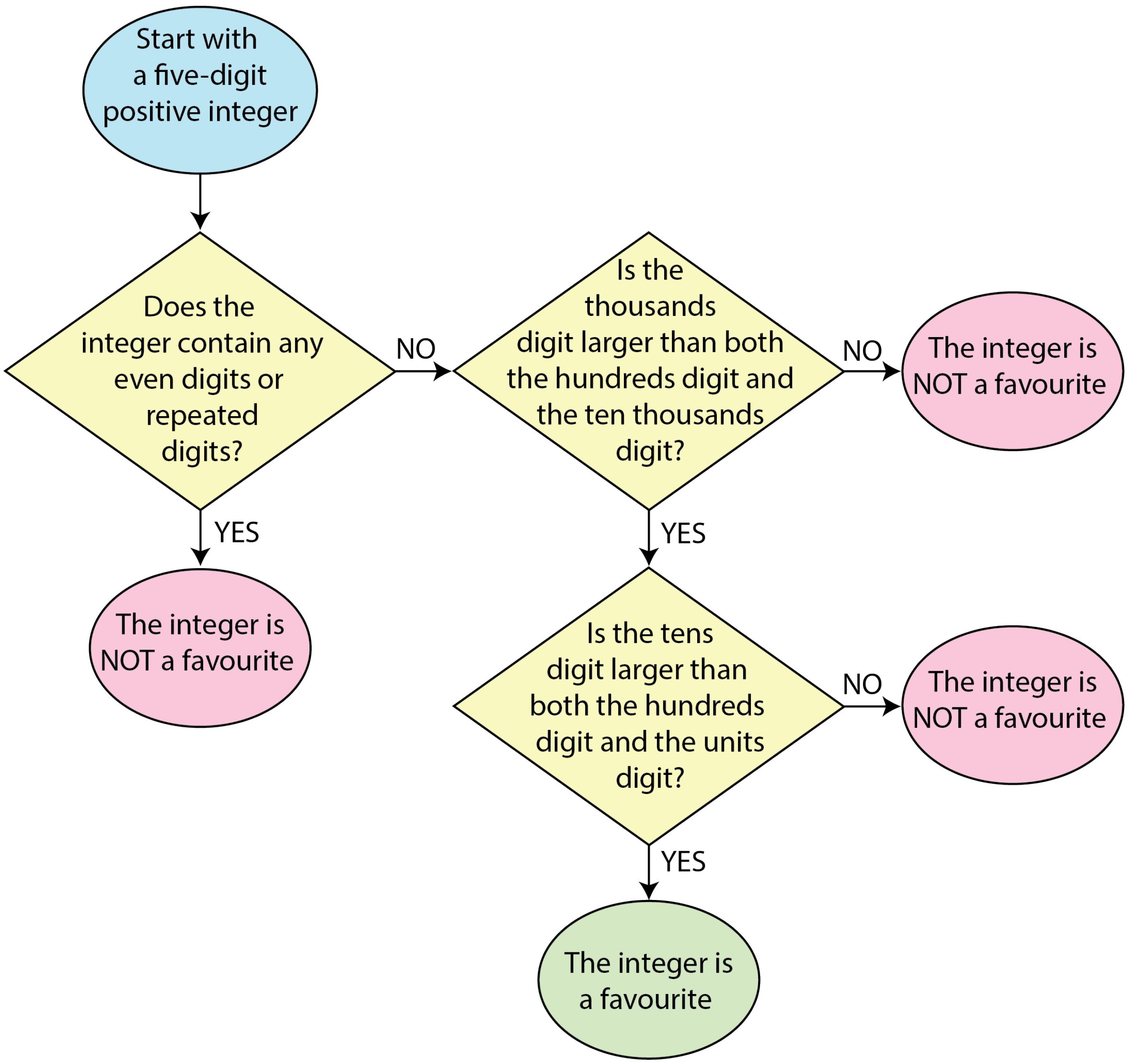
$$\begin{aligned} 0.3(96 + (z + 12)) &= (z + 12) + z \\ 28.8 + 0.3z + 3.6 &= 2z + 12 \\ 20.4 &= 1.7z \\ z &= 12 \end{aligned}$$

Therefore, there are 12 students who do not like pineapples and do not like bananas.

Although we are not asked to do so, we could go on and solve for $x = 24$ and $y = 24$.

Problem of the Week
Problem E
Favourite Numbers

Adrian likes all the numbers, but some are his favourites. He created a flowchart to help people determine whether or not a given five-digit positive integer is one of his favourites.

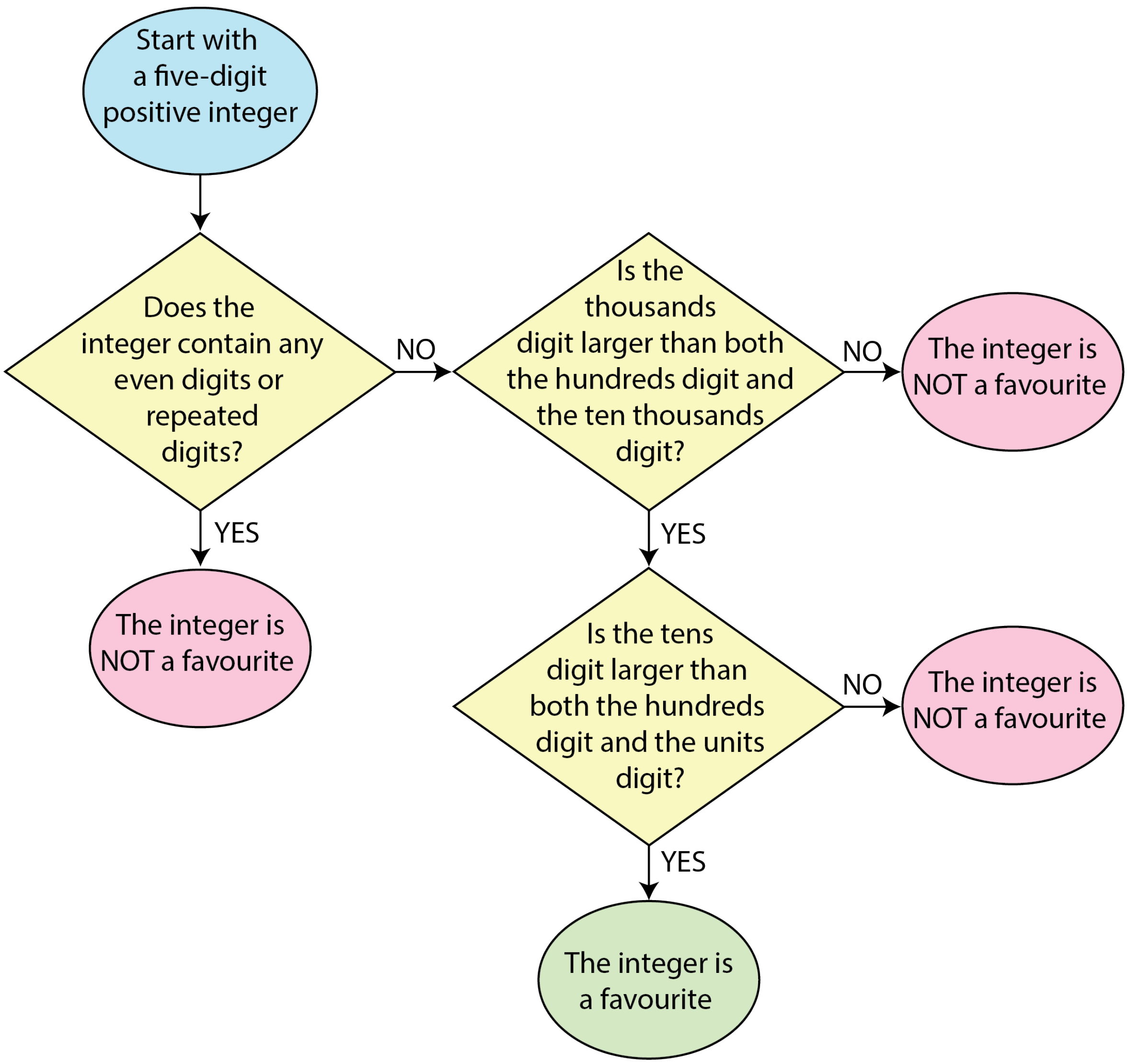


How many favourite five-digit positive integers does Adrian have?

Problem of the Week
Problem E and Solution
Favourite Numbers

Problem

Adrian likes all the numbers, but some are his favourites. He created a flowchart to help people determine whether or not a given five-digit positive integer is one of his favourites.



How many favourite five-digit positive integers does Adrian have?

Solution

We write one of Adrian's favourite five-digit positive integers as $VWXYZ$, where each letter represents a digit.

Since this integer does not contain any even or repeated digits, then it is created using the digits $1,\ 3,\ 5,\ 7$, and 9, in some order. We want to count the number of ways of assigning $1, 3, 5, 7, 9$ to the digits V, W, X, Y, Z so that the answers to the second two questions in the flowchart are both yes.

Since the thousands digit is larger than both the hundreds digit and the ten thousands digit, then $W > X$ and $W > V$. Since the tens digit is larger than both the hundreds digit and the units digit, then $Y > X$ and $Y > Z$.

The digits 1 and 3 cannot be placed as W or Y, since W and Y are larger than both of their neighbouring digits, while 1 is smaller than all of the other digits and 3 is larger than only one of the other possible digits.

The digit 9 cannot be placed as V, X, or Z since it is the largest possible digit and so cannot be smaller than W or Y . Thus, 9 must be placed as W or as Y. Therefore, the digits W and Y are 9 and either 5 or 7.

Suppose that $W = 9$ and $Y = 5$. The number is thus $V9X5Z$. Neither X or Z can equal 7 since $7 > 5$, so $V = 7$. It follows that X and Z are 1 and 3, or 3 and 1. There are 2 possible integers in this case. Similarly, if $Y = 9$ and $W = 5$, there are 2 possible integers.

Suppose that $W = 9$ and $Y = 7$. The number is thus $V9X7Z$. The digits $1, 3$, and 5 can then be placed in any of the remaining spots. There are 3 choices for the digit V. For each of these choices, there are 2 choices for X, and then 1 choice for Z. There are thus $3 \times 2 \times 1 = 6$ possible integers in this case. Similarly, if $Y = 9$ and $W = 7$, there are 6 possible integers.

Therefore, Adrian has $2 + 2 + 6 + 6 = 16$ favourite positive five-digit integers.

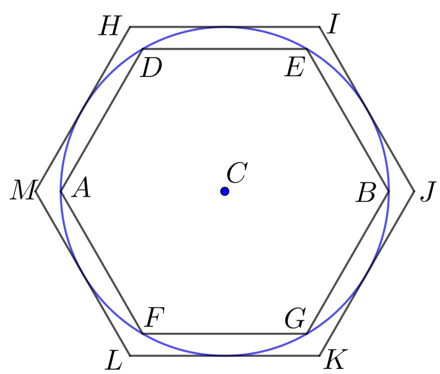
Problem of the Week
Problem E
Pi Hexagons

Pi Day is an annual celebration of the mathematical constant π. Pi Day is observed on March 14, since 3, 1, and 4 are the first three significant digits of π.

Archimedes determined lower bounds for π by finding the perimeters of regular polygons inscribed in a circle with diameter of length 1. (An inscribed polygon of a circle has all of its vertices on the circle.) He also determined upper bounds for π by finding the perimeters of regular polygons circumscribed in a circle with diameter of length 1. (A circumscribed polygon of a circle has all sides tangent to the circle. That is, each side of the polygon touches the circle in one spot.)

In this problem, we will determine a lower bound for π and an upper bound for π by considering an inscribed regular hexagon and a circumscribed regular hexagon in a circle of diameter 1.

Consider a circle with centre C and diameter 1. Since the circle has diameter 1, it has circumference equal to π. Now consider the inscribed regular hexagon $DEBGFA$ and the circumscribed regular hexagon $HIJKLM$.

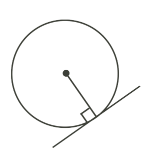


The perimeter of hexagon $DEBGFA$ will be less than the circumference of the circle, π, and will thus give us a lower bound for the value of π. The perimeter of hexagon $HIJKLM$ will be greater than the circumference of the circle, π, and will thus give us an upper bound for the value of π.

Using these hexagons, determine a lower and an upper bound for π.

NOTE: For this problem, you may want to use the following known results:

1. A line drawn from the centre of a circle perpendicular to a tangent line meets the tangent line at the point of tangency.



2. For a circle with centre C, the centres of both the inscribed and circumscribed regular hexagons will be at C.

Problem of the Week
Problem E and Solution
Pi Hexagons

Problem

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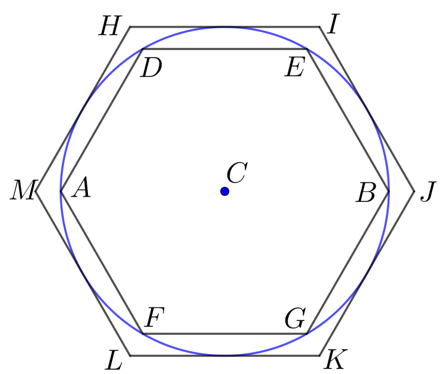

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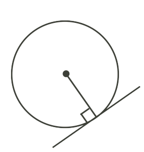
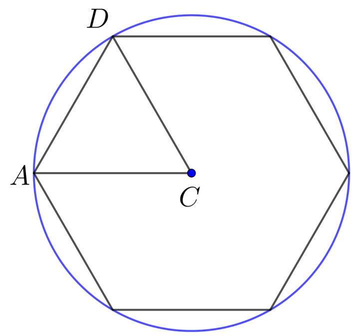


2. For a circle with centre C, the centres of both the inscribed and circumscribed regular hexagons will be at C.

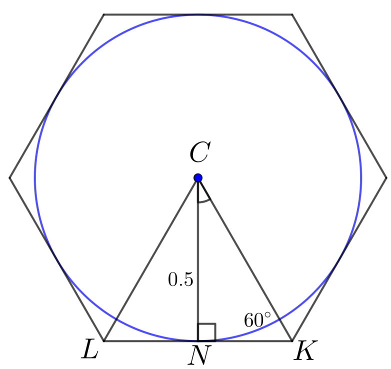
Solution

For the inscribed hexagon, draw line segments AC and DC, which are both radii of the circle.



Since the diameter of the circle is 1, $AC = DC = \frac{1}{2}$. Since the inscribed hexagon is a regular hexagon with centre C, we know that $\triangle ACD$ is equilateral (a justification of this is provided at the end of the solution). Thus, $AD = AC = \frac{1}{2}$, and the perimeter of the inscribed regular hexagon is $6 \times AD = 6\left(\frac{1}{2}\right) = 3$. Since the perimeter of this hexagon is less than the circumference of the circle, this gives us a lower bound for π. That is, this tells us that $\pi > 3$.

For the circumscribed hexagon, draw line segments LC and KC. Since the circumscribed hexagon is a regular hexagon with centre C, we know that $\triangle LCK$ is equilateral (a justification of this is provided at the end of the solution). Thus, $\angle LKC = 60°$. Drop a perpendicular from C, meeting LK at N. We know that N must be the point of tangency. Thus, CN is a radius and so $CN = 0.5$. In $\triangle CNK$, $\angle NKC = \angle LKC = 60°$.



Since $\angle CNK = 90°$,

$$\begin{aligned}\sin(\angle NKC) &= \frac{CN}{KC}\\ \sin(60°) &= \frac{0.5}{KC}\\ \frac{\sqrt{3}}{2} &= \frac{0.5}{KC}\\ \sqrt{3}KC &= 1\\ KC &= \frac{1}{\sqrt{3}}\end{aligned}$$

But $\triangle LCK$ is equilateral, so $LK = KC = \frac{1}{\sqrt{3}}$.

Thus, the perimeter of the circumscribed hexagon is $6 \times LK = 6 \times \frac{1}{\sqrt{3}} = \frac{6}{\sqrt{3}} \approx 3.46$.

Since the perimeter of this hexagon is greater than the circumference of the circle, this gives us an upper bound for π. That is, this tells us that $\pi < \frac{6}{\sqrt{3}}$.

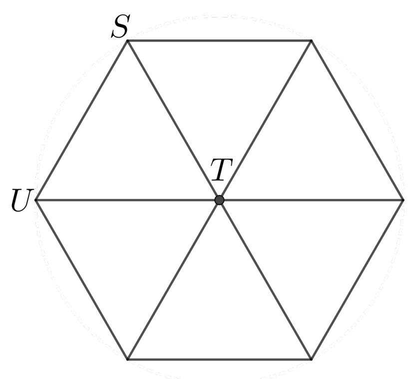
Therefore, the value for π is between 3 and $\frac{6}{\sqrt{3}}$. That is, $3 < \pi < \frac{6}{\sqrt{3}}$.

Extension: Archimedes used regular 12-gons, 24-gons, 48-gons and 96-gons to get better approximations for the bounds on π. Can you?

Equilateral triangle justification:

In the solutions, we used the fact that both $\triangle ACD$ and $\triangle LCK$ are equilateral. In fact, a regular hexagon can be split into six equilateral triangles by drawing line segments from the centre of the hexagon to each vertex, which we will now justify.

Consider a regular hexagon with centre T. Draw line segments from T to each vertex and label two adjacent vertices S and U.



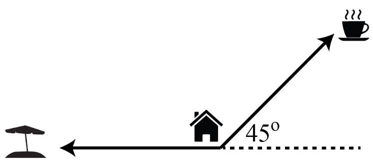
Since T is the centre of the hexagon, T is of equal distance to each vertex of the hexagon. Since the hexagon is a regular hexagon, each side of the hexagon has equal length. Thus, the six resultant triangles are congruent. Therefore, the six central angles are equal and each is equal to $\frac{1}{6}(360^\circ) = 60^\circ$.

Now consider $\triangle STU$. We know that $\angle STU = 60^\circ$. Also, $ST = UT$, so $\triangle STU$ is isosceles and $\angle TSU = \angle TUS = \frac{180^\circ - 60^\circ}{2} = 60^\circ$.

Therefore, all three angles in $\triangle STU$ are equal to 60° and so $\triangle STU$ is equilateral. Since the six triangles in the hexagon are congruent, this tells us that all six triangles are all equilateral.

Problem of the Week
Problem E
Coffee Run

On Saturday morning at 8 a.m., Ayla and Hamza left their house. Ayla walked west towards the beach, and Hamza rode his e-scooter northeast to his favourite coffee shop, then headed to catch up with Ayla. Ayla walked at a constant speed of 4 km/h and Hamza rode his e-scooter at a constant speed of 20 km/h. If Hamza caught up with Ayla exactly 45 minutes after they left their house, what is the maximum possible distance between their house and the coffee shop?



Problem of the Week

Problem E and Solution

Coffee Run

Problem

On Saturday morning at 8 a.m., Ayla and Hamza left their house. Ayla walked west towards the beach, and Hamza rode his e-scooter northeast to his favourite coffee shop, then headed to catch up with Ayla. Ayla walked at a constant speed of 4 km/h and Hamza rode his e-scooter at a constant speed of 20 km/h. If Hamza caught up with Ayla exactly 45 minutes after they left their house, what is the maximum possible distance between their house and the coffee shop?

Solution

To find the maximum possible distance between their house and the coffee shop, we will assume that Hamza traveled in a straight line from his house to the coffee shop, and also from the coffee shop to catch up with Ayla. We will also assume that Hamza spent no time at the coffee shop. In reality these assumptions are unlikely, however they are necessary to determine the maximum possible distance.

Solution 1

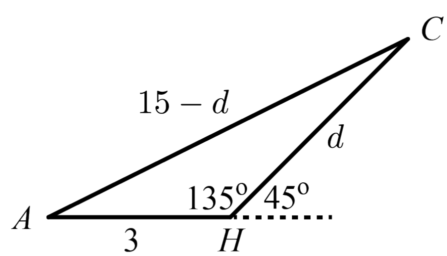
We know that the total time is 45 minutes, or $\frac{3}{4}$ of an hour, and in that time Ayla walked $\frac{3}{4} \times 4 = 3$ km.

Let t represent the time, in hours, that it took Hamza to travel to the coffee shop. Then, he took $\left(\frac{3}{4} - t\right)$ hours to travel from the coffee shop to meet up with Ayla. The distance between their house and the coffee shop is then $20t$ km, and the distance between the coffee shop and the point where Hamza met up with Ayla is $20\left(\frac{3}{4} - t\right) = (15 - 20t)$ km.

On the diagram, H represents their home, C represents the coffee shop, and A represents the point where Hamza met up with Ayla. We can determine that $\angle CHA = 180^\circ - 45^\circ = 135^\circ$. So $CH = 20t$ km, $AH = 3$ km, and $AC = (15 - 20t)$ km. If we let $d = 20t$, we can simplify CH to d and AC to $15 - d$.

Using the cosine law,

$$\begin{aligned}
AC^2 &= AH^2 + CH^2 - 2(AH)(CH)\cos(\angle CHA) \\
(15-d)^2 &= 3^2 + d^2 - 2(3)(d)\cos 135^\circ \\
225 - 30d + d^2 &= 9 + d^2 - 6d\left(-\frac{1}{\sqrt{2}}\right) \\
216 &= \frac{6d}{\sqrt{2}} + 30d \\
d &= 216 \div \left(\frac{6}{\sqrt{2}} + 30\right) \approx 6.3 \text{ km}
\end{aligned}$$



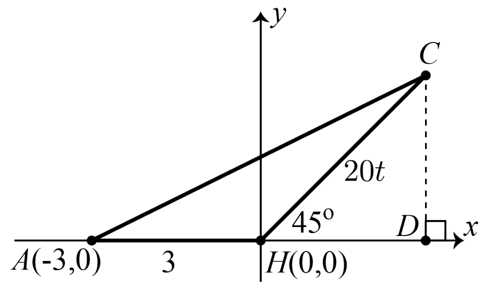
Therefore, the maximum possible distance between their house and the coffee shop is $216 \div \left(\frac{6}{\sqrt{2}} + 30\right) \approx 6.3$ km.

Solution 2

As in Solution 1, we know that the total time is 45 minutes, or $\frac{3}{4}$ of an hour, and in that time Ayla walked $\frac{3}{4} \times 4 = 3$ km. We will describe the positions of Ayla and Hamza in terms of points in the coordinate plane. Let the point $H(0,0)$ represent their home. Then Ayla walked along the negative x-axis, and the point at which she met up with Hamza is $A(-3,0)$.

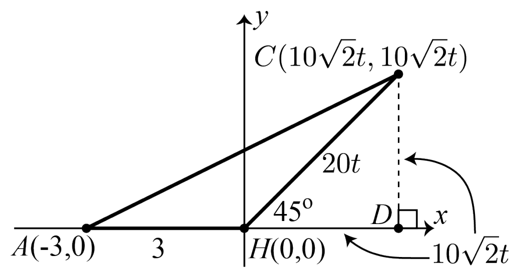
Let t represent the time, in hours, that it took Hamza to travel to the coffee shop, and let C represent the coffee shop. Then the length of CH is $20t$ km. To determine the coordinates of C, we will draw a vertical line from C that meets the x-axis at point D. Since $\angle CHD = 45^\circ$, it follows that $\triangle CDH$ is an isosceles right-angled triangle.

$$\begin{aligned} \sin 45^\circ &= \frac{CD}{20t} \\ \frac{1}{\sqrt{2}} &= \frac{CD}{20t} \\ CD &= \frac{20t}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{20\sqrt{2}t}{2} = 10\sqrt{2}t \end{aligned}$$



Since $\triangle CDH$ is an isosceles right-angled triangle, $HD = 10\sqrt{2}t$. Thus, the coordinates of C are $(10\sqrt{2}t, 10\sqrt{2}t)$. The distance between A and C, d_{AC}, can then be calculated.

$$\begin{aligned} (d_{AC})^2 &= (10\sqrt{2}t - (-3))^2 + (10\sqrt{2}t - 0)^2 \\ &= 100(2)t^2 + 60\sqrt{2}t + 9 + 100(2)t^2 \\ &= 400t^2 + 60\sqrt{2}t + 9 \\ d_{AC} &= \sqrt{400t^2 + 60\sqrt{2}t + 9} \end{aligned}$$



Since Hamza traveled at 20 km/h, the time it took him to travel this distance was $\frac{\sqrt{400t^2+60\sqrt{2}t+9}}{20}$ hours. It took Hamza $\frac{3}{4}$ of an hour to travel from H to C and then from C to A. Thus,

$$\begin{aligned} t + \frac{\sqrt{400t^2 + 60\sqrt{2}t + 9}}{20} &= \frac{3}{4} \\ 20t + \sqrt{400t^2 + 60\sqrt{2}t + 9} &= 15 \\ \sqrt{400t^2 + 60\sqrt{2}t + 9} &= 15 - 20t \\ 400t^2 + 60\sqrt{2}t + 9 &= (15 - 20t)^2 \\ 400t^2 + 60\sqrt{2}t + 9 &= 225 - 600t + 400t^2 \\ 600t + 60\sqrt{2}t &= 216 \\ t &= \frac{216}{600 + 60\sqrt{2}} \text{ hours} \end{aligned}$$

It follows that $HC = 20t = 20\left(\frac{216}{600+60\sqrt{2}}\right) = \frac{216}{30+3\sqrt{2}} \approx 6.3$ km. Therefore, the maximum possible distance between their house and the coffee shop is $\frac{216}{30+3\sqrt{2}} \approx 6.3$ km.

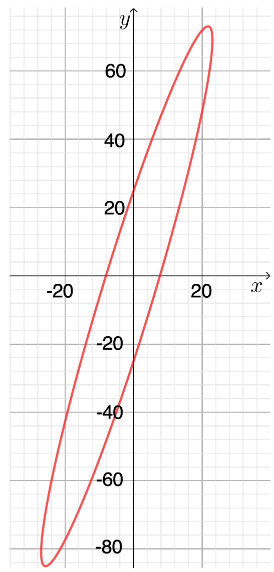
Problem of the Week

Problem E

Points on an Ellipse

The graph of $(x+1)^2 + (y-2)^2 = 100$ is a circle with centre $(-1, 2)$ and radius 10.

The graph of $10x^2 - 6xy + 4x + y^2 = 621$ is shown below. The shape of this curve is known as an ellipse.



List all the ordered pairs (x, y) of non-negative integers x and y that satisfy the equation $10x^2 - 6xy + 4x + y^2 = 621$.

Note: When solving this problem, it might be useful to use the following idea.

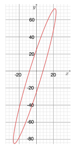
By completing the square,

$$x^2 + y^2 + 2x - 4y = 95$$

can be rewritten as

$$(x+1)^2 + (y-2)^2 = 100$$

One solution to this equation is $(x, y) = (5, 10)$.



Problem of the Week

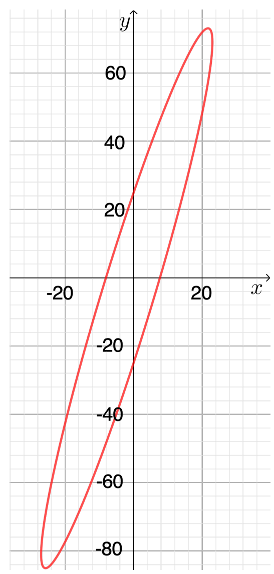
Problem E and Solution

Points on an Ellipse

Problem

The graph of $(x+1)^2 + (y-2)^2 = 100$ is a circle with centre $(-1, 2)$ and radius 10.

The graph of $10x^2 - 6xy + 4x + y^2 = 621$ is shown below. The shape of this curve is known as an ellipse.



List all the ordered pairs (x, y) of non-negative integers x and y that satisfy the equation $10x^2 - 6xy + 4x + y^2 = 621$.

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By completing the square,

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can be rewritten as

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Solution

Starting with the given equation, we obtain the following equivalent equations:

$$10x^2 - 6xy + 4x + y^2 = 621$$
$$9x^2 - 6xy + y^2 + x^2 + 4x = 621$$
$$9x^2 - 6xy + y^2 + x^2 + 4x + 4 = 621 + 4$$
$$(3x - y)^2 + (x + 2)^2 = 625$$

Notice that $625 = 25^2$.
Since x and y are both integers, then the left side of the given equation is the sum of two perfect squares. Since any perfect square is non-negative, then each of these perfect squares is at most $625 = 25^2$.

The pairs of perfect squares that sum to 625 are 625 and 0, 576 and 49, and 400 and 225.

Therefore, $(3x - y)^2$ and $(x + 2)^2$ are equal to 25^2 and 0^2 in some order, or 24^2 and 7^2 in some order, or 20^2 and 15^2 in some order.

Furthermore, $3x - y$ and $x + 2$ are equal to ± 25 and ± 0 in some order, or ± 24 and ± 7 in some order, or ± 20 and ± 15 in some order.

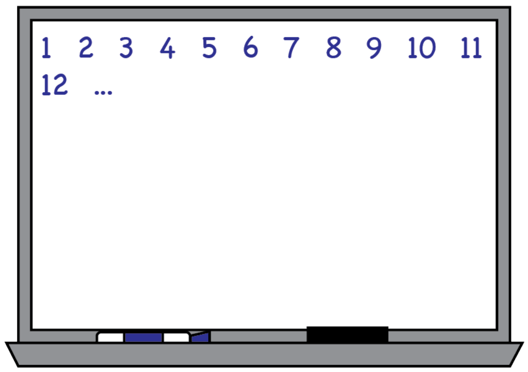
Since $x \geq 0$, then $x + 2 \geq 2$. So we need to consider when $x + 2$ is equal to 25, 24, 7, 20, or 15.

- If $x + 2 = 25$, then $x = 23$. Also, $3x - y = 0$. Thus, $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(23, 69)$ is a valid ordered pair.
- If $x + 2 = 24$, then $x = 22$. Also, $3x - y = 7$ or $3x - y = -7$.
  When $3x - y = 7$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(22, 59)$ is a valid ordered pair.
  When $3x - y = -7$, we find $y = 73$. Since $x \geq 0$ and $y \geq 0$, $(22, 73)$ is a valid ordered pair.
- If $x + 2 = 7$, then $x = 5$. Also, $3x - y = 24$ or $3x - y = -24$.
  When $3x - y = 24$, we find $y = -9$. Since $y < 0$, this does not lead to a valid ordered pair.
  When $3x - y = -24$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(5, 39)$ is a valid ordered pair.
- If $x + 2 = 20$, then $x = 18$. Also, $3x - y = 15$ or $3x - y = -15$.
  When $3x - y = 15$, we find $y = 39$. Since $x \geq 0$ and $y \geq 0$, $(18, 39)$ is a valid ordered pair.
  When $3x - y = -15$, we find $y = 69$. Since $x \geq 0$ and $y \geq 0$, $(18, 69)$ is a valid ordered pair.
- If $x + 2 = 15$, then $x = 13$. Also, $3x - y = 20$ or $3x - y = -20$.
  When $3x - y = 20$, we find $y = 19$. Since $x \geq 0$ and $y \geq 0$, $(13, 19)$ is a valid ordered pair.
  When $3x - y = -20$, we find $y = 59$. Since $x \geq 0$ and $y \geq 0$, $(13, 59)$ is a valid ordered pair.

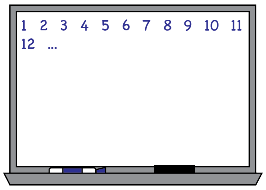
Therefore, the ordered pairs of non-negative integers that satisfy the equation are $(23, 69)$, $(22, 59)$, $(22, 73)$, $(5, 39)$, $(18, 39)$, $(18, 69)$, $(13, 19)$, and $(13, 59)$.

Problem of the Week
Problem E
Wipe Away 3

Tyra writes consecutive positive integers on a whiteboard starting with the integer 1. However, when she writes a number that is a multiple of 9, or contains the digit 9, Juliana immediately erases it. If they continue this for a long time, what is the 400$^{\text{th}}$ number that Juliana will erase?



NOTE: In solving this problem, it may be helpful to use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9. For example, the number 214 578 is divisible by 9 since $2+1+4+5+7+8=27$, which is divisible by 9. In fact, $214\,578 = 9 \times 23\,842$.



Problem of the Week
Problem E and Solution
Wipe Away 3

Problem

Tyra writes consecutive positive integers on a whiteboard starting with the integer 1. However, when she writes a number that is a multiple of 9, or contains the digit 9, Juliana immediately erases it. If they continue this for a long time, what is the 400^{th} number that Juliana will erase?

NOTE: In solving this problem, it may be helpful to use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9. For example, the number 214 578 is divisible by 9 since $2 + 1 + 4 + 5 + 7 + 8 = 27$, which is divisible by 9. In fact, $214\,578 = 9 \times 23\,842$.

Solution

We first consider the integers between 1 and 999, inclusive. Since $999 = 111 \times 9$, there are 111 multiples of 9 between 1 and 999.

Now let's figure out how many of the integers from 1 to 999 contain the digit 9. The integers from 1 to 99 that contain the digit 9 are $9,\ 19, \ldots, 79,\ 89$ as well as $90,\ 91, \ldots, 97\ , 98,\ 99$. Thus, there are 19 positive integers from 1 to 99 that contain the digit 9. Since there are 19 integers from 1 to 99 that contain the digit 9, it follows that there are $19 \times 9 = 171$ integers from 1 to 899 that contain the digit 9.

Between 900 and 999, every integer contains the digit 9. Thus, there are 100 numbers that contain the digit 9. Thus, in total, $171 + 100 = 271$ of the integers from 1 to 999 contain the digit 9.

However, some of the integers that contain the digit 9 are also multiples of 9, so were counted twice. To determine how many of these such numbers there are, we use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9.

- The only one-digit number that contains the digit 9 and is also a multiple of 9 is 9 itself.
- The only two-digit numbers that contain the digit 9 and are also multiples of 9 are 90 and 99.
- To find the three-digit numbers that contain the digit 9 and are also multiples of 9, we will look at their digit sum.
  - **Case 1:** Three digit-numbers with a digit sum of 9:
    The only possibility is 900. Thus, there is 1 number.
  - **Case 2:** Three digit-numbers with a digit sum of 18:
    * If two of the digits are 9, then the other digit must be 0. The only possibilities are 909 and 990. Thus, there are 2 numbers.
    * If only one of the digits is 9, then the other two digits must add to 9. The possible digits are $9,\ 4,\ 5$, or $9,\ 3,\ 6$, or $9,\ 2,\ 7$, or $9,\ 8,\ 1$. For each of these sets of digits, there are 3 choices for the hundreds digit. Once the hundreds

digit is chosen, there are 2 choices for the tens digit, and then the remaining digit must be the ones digit. Thus, there are $3 \times 2 = 6$ possible three-digit numbers for each set of digits. Since there are 4 sets of digits, then there are $4 \times 6 = 24$ possible numbers.

- **Case 3:** Three digit-numbers with a digit sum of 27:
  The only possibility is 999. Thus, there is 1 number.

Therefore, there are $1 + 2 + 24 + 1 = 28$ three-digit numbers from 1 to 999 that contain the digit 9, and are also multiples of 9.

Thus, there are $1 + 2 + 28 = 31$ integers from 1 to 999 that contain the digit 9, and are also multiples of 9. It follows that Juliana erases $111 + 271 - 31 = 351$ of the numbers from 1 to 999 from the whiteboard. Since we are looking for the 400^{th} number that Juliana erases, we need to keep going.

Next, we consider the integers between 1000 and 1099, inclusive. Since $1099 = (122 \times 9) + 1$, there are 122 multiples of 9 between 1 and 1099. Since there are 111 multiples of 9 between 1 and 999, it follows that there are $122 - 111 = 11$ multiples of 9 between 1000 and 1099. The integers between 1000 and 1099 that contain the digit 9 are $1009,\ 1019, \ldots, 1079,\ 1089$ as well as $1090,\ 1091, \ldots, 1097\ , 1098,\ 1099$. Thus, there are 19 integers from 1000 to 1099 that contain the digit 9. Of these, the only integers that are also multiples of 9 are 1089 and 1098. Thus, Juliana erases $11 + 19 - 2 = 28$ of the numbers from 1000 to 1099 from the whiteboard. In total, she has now erased $351 + 28 = 379$ numbers.

Next, we consider the integers between 1100 and 1189, inclusive. Since $1189 = (132 \times 9) + 1$, there are 132 multiples of 9 between 1 and 1189. Since there are 122 multiples of 9 between 1 and 1099, it follows that there are $132 - 122 = 10$ multiples of 9 between 1100 and 1189. The integers between 1100 and 1189 that contain the digit 9 are $1109,\ 1119, \ldots, 1179,\ 1189$. Thus there are 9 integers from 1100 to 1189 that contain the digit 9. The only one of these that is also a multiple of 9 is 1179. Thus, Juliana erases $10 + 9 - 1 = 18$ of the numbers from 1100 to 1189 from the whiteboard. In total, she has now erased $379 + 18 = 397$ numbers.

The next three numbers that Juliana will erase are $1190,\ 1191$, and 1192. Thus, the 400^{th} number that Juliana erases is 1192.