



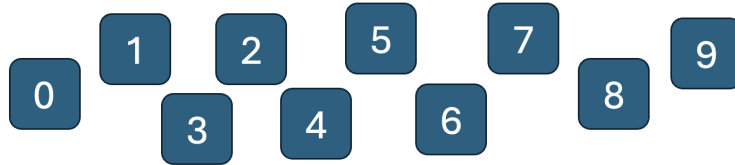
Problem of the Week

Problem D and Solution

Counting with Digit Tiles

Problem

For each digit from 0 to 9, you have 100 identical tiles with that digit on it.



Using these 1000 tiles, you start building consecutive positive integers, starting with 1. Each time you build an integer, you must remove the digit tiles you used from your stockpile of tiles.

For example, after you have built the integers from 1 to 13, the table below summarizes how many tiles remain for each digit.

Digit	0	1	2	3	4	5	6	7	8	9
Number of Tiles Remaining	99	94	98	98	99	99	99	99	99	99

What is the largest integer you can build without running out of the tiles needed to form the integer?

Solution

For the integers 1 to 99, each digit from 1 to 9 will be used the same number of times. The digit 0 will be used less than every other digit. As soon as we start to build the integer 100, the digit 1 will be used more frequently than any other digit. So, we will first run out of tiles with the digit 1 on them. Thus, we will count the number of times we can use a tile with the digit 1.

From 1 to 99, there is a 1 in the units digit of integers 1, 11, 21, 31, 41, 51, 61, 71, 81, 91, a total of 10 integers. There are also 10 integers with tens digit 1: 10, 11, 12, 13, 14, 15, 16, 17, 18, and 19. Thus, by the time we reach 99, we have used $10 + 10 = 20$ tiles with the digit 1.

From 100 to 109, we use the digit 1 in the hundreds position 10 times, and in the units position of integer 101. Thus, we use 11 more tiles with the digit 1, so we have now used $20 + 11 = 31$ such tiles.

From 110 to 119, we use the digit 1 in the hundreds position 10 times, in the tens position 10 times, and once in the units position. This makes a total of 21 more tiles with the digit 1 used, so we have now used $31 + 21 = 52$ such tiles.

For each of 120 to 129, 130 to 139, 140 to 149, and 150 to 159, we use the same number of tiles with the digit 1 as we did when building integers from 100 to 109. So, to build the integers from 120 to 159 we use $11 + 11 + 11 + 11 = 44$ tiles with the digit 1, and have now used a total of $52 + 44 = 96$ such tiles.

Now we have four tiles with the digit 1 remaining. We can build 160, 161, and 162. After we have built the integer 162, no more tiles with the digit 1 remain, and so we cannot build the integer 163.

Therefore, the largest integer we can build to before running out of the necessary digits to create the integer is 162.