Problem of the Week

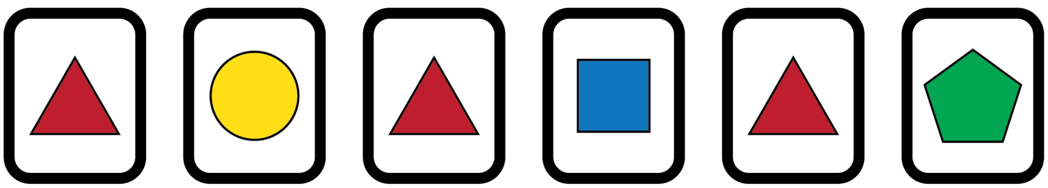
Problem D and Solution

Six Cards

Problem

Antonia has a set of cards where each card has a shape on one side and a digit from 0 to 9 on the other side. Any two cards with the same shape have the same digit on the other side, and any two cards with different shapes have different digits on the other side.

Antonia lays out the following six cards.



She then flips each card over in place and records the six-digit number they form. For example, if there is a 4 on the other side of the cards with a triangle, a 2 on the other side of the card with a circle, a 7 on the other side of the card with a square, and a 5 on the other side of the card with a pentagon, then the six-digit number they form would be 424 745.

Antonia notices that the six-digit number they form is divisible by 11. Determine the largest and smallest possible six-digit numbers that this could be.

NOTE: You may find the following fact useful:

A number is divisible by 11 exactly when the sum of the digits in the odd digit positions minus the sum of the digits in the even digit positions is divisible by 11. For example, the number 138 248 is divisible by 11 since $(1 + 8 + 4) - (3 + 2 + 8) = 13 - 13 = 0$ and 0 is divisible by 11. The number 693 748 is also divisible by 11 since $(6 + 3 + 4) - (9 + 7 + 8) = 13 - 24 = -11$ and -11 is divisible by 11.

Solution

Let T represent the digit on the other side of the cards with the triangle, let C represent the digit on the other side of the card with the circle, let S represent the digit on the other side of the card with the square, and let P represent the digit on the other side of the card with the pentagon. Then the six-digit number can be represented as $TCT\,STP$.

We will start by finding the largest possible six-digit number. To do this, we will make the leftmost digit as large as possible. So $T = 9$. Then our six-digit number is $9C9\,S9P$. Next we will make C as large as possible, so $C = 8$. Then our six-digit number is $989\,S9P$. Next we will make S as large as possible, so $S = 7$. Then our six-digit number is $989\,79P$.

If $989\,79P$ is divisible by 11, then $(9+9+9)-(8+7+P) = 27-15-P = 12-P$ is also divisible by 11. The only possible single-digit value for P is $P = 1$, and we have not already used this digit. Since $989\,791 = 11 \times 89\,981$, we can verify that $989\,791$ is divisible by 11. Thus, the largest possible six-digit number is $989\,791$.

Next we will find the smallest possible six-digit number. To do this, we will make the leftmost digit as small as possible. So $T = 1$. Note that it's not possible for T to equal 0, because we need to have a six-digit number. Then our six-digit number is $1C1\,S1P$. Next we will make C as small as possible, so $C = 0$. Then our six-digit number is $101\,S1P$. Next we will make S as small as possible, so $S = 2$. Then our six-digit number is $101\,21P$.

If $101\,21P$ is divisible by 11, then $(1+1+1)-(0+2+P) = 3-2-P = 1-P$ is also divisible by 11. The only possible single-digit value for P is $P = 1$, however we already set $T = 1$. So we must try a larger value for S.

We will try the next smallest possible value for S, $S = 3$. Then our six-digit number is $101\,31P$. If $101\,31P$ is divisible by 11, then $(1+1+1)-(0+3+P) = 3-3-P = -P$ is also divisible by 11. The only possible single-digit value for P is $P = 0$, however we already set $C = 0$. So we must try a larger value for S.

We will try $S = 4$. Then our six-digit number is $101\,41P$. If $101\,41P$ is divisible by 11, then $(1+1+1)-(0+4+P) = 3-4-P = -1-P$ is also divisible by 11. There is no possible value for P that is a single digit, so we must try a larger value for S.

We will try $S = 5$. Then our six-digit number is $101\,51P$. If $101\,51P$ is divisible by 11, then $(1+1+1)-(0+5+P) = 3-5-P = -2-P$ is also divisible by 11. The only possible single-digit value for P is $P = 9$, and we have not already used this digit. Since $101\,519 = 11 \times 9229$, we can verify that $101\,519$ is divisible by 11. Thus, the smallest possible six-digit number is $101\,519$.