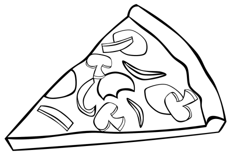

Problem of the Week

Problem D and Solution

Ten Slice Pizza

Problem

The DECI-Pizza Company has a special pizza that has 10 slices. Two of the slices are each $\frac{1}{6}$ of the whole pizza, two are each $\frac{1}{8}$, four are each $\frac{1}{12}$, and two are each $\frac{1}{24}$. A group of n friends share the pizza by distributing all of these slices. They do not cut any of the slices. Each of the n friends receives, in total, an equal fraction of the whole pizza. For what values of $n > 1$ is this possible?

Solution

Solution 1

Each of the n friends is to receive $\frac{1}{n}$ of the pizza.

Since there are two slices that are each $\frac{1}{6}$ of the pizza and these slices cannot be cut, then each friend receives at least $\frac{1}{6}$ of the pizza. This means that there cannot be more than 6 friends. That is, $n \leq 6$.

The value $n = 2$ is possible. We show this by dividing the slices into two groups, each of which totals $\frac{1}{2}$ of the pizza. Note that $\frac{1}{6} + \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{2}$. This means the other six slices must also add to $\frac{1}{2}$.

The value $n = 3$ is possible. We show this by finding three groups of slices, with each group totaling $\frac{1}{3}$ of the pizza. Since $2 \times \frac{1}{6} = \frac{1}{3}$ and $4 \times \frac{1}{12} = \frac{1}{3}$, then the other four slices must also add to $\frac{1}{3}$ (the rest of the pizza), and so $n = 3$ is possible.

The value $n = 4$ is possible since $2 \times \frac{1}{8} = \frac{1}{4}$ and $\frac{1}{6} + \frac{1}{12} = \frac{1}{4}$ (which can be done twice). The other four slices must also add to $\frac{1}{4}$.

The value $n = 6$ is possible since two slices are $\frac{1}{6}$ on their own, two groups of size $\frac{1}{6}$ can be made from the four slices of size $\frac{1}{12}$, and $\frac{1}{8} + \frac{1}{24} = \frac{1}{6}$ (which can be done twice), which makes six groups of size $\frac{1}{6}$.

The value $n = 5$ is not possible, since to make a portion of size $\frac{1}{5}$ that includes a slice of size $\frac{1}{6}$, the remaining slices must total $\frac{1}{5} - \frac{1}{6} = \frac{1}{30}$. Since every slice is larger than $\frac{1}{30}$, this is not possible.

Therefore, the possible values of n are 2, 3, 4, and 6.

Solution 2

The pizza is cut into two slices of size $\frac{1}{24}$, four slices of size $\frac{1}{12}$, four slices of size $\frac{1}{8}$, and two slices of size $\frac{1}{6}$.

Each of these fractions can be written with a denominator of 24. Thus, this is equivalent to saying there are two slices of size $\frac{1}{24}$, four slices of size $\frac{2}{24}$, two slices of size $\frac{3}{24}$, and two slices of size $\frac{4}{24}$.

To create groups of slices of equal total size, we can now consider combining the integers 1, 1, 2, 2, 2, 2, 3, 3, 4, and 4 into groups with equal sum. (These integers represent the size of each slice measured in units of $\frac{1}{24}$ of the pizza.)

Since the largest integer in the list is 4, then each group has to have size at least 4. Since $4 = 24 \div 6$, then the slices cannot be broken into more than 6 groups of equal size, which means that n cannot be greater than 6.

Here is a way of breaking the slices into $n = 6$ equal groups, each with total size $24 \div 6 = 4$:

$$4 \qquad 4 \qquad 3+1 \qquad 3+1 \qquad 2+2 \qquad 2+2$$

Here is a way of breaking the slices into $n = 4$ equal groups, each with total size $24 \div 4 = 6$:

$$4+2 \qquad 4+2 \qquad 3+3 \qquad 2+2+1+1$$

Here is a way of breaking the slices into $n = 3$ equal groups, each with total size $24 \div 3 = 8$:

$$4+4 \qquad 2+2+2+2 \qquad 3+3+1+1$$

Here is a way of breaking the slices into $n = 2$ equal groups, each with total size $24 \div 2 = 12$:

$$4+4+2+2 \qquad 3+3+2+2+1+1$$

Since 24 is not a multiple of 5, the slices cannot be broken into $n = 5$ groups of equal size.

Therefore, the possible values of n are 2, 3, 4, and 6.