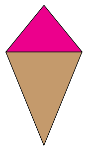


Problem of the Week

Problem D and Solution

Who Wants Ice Cream?

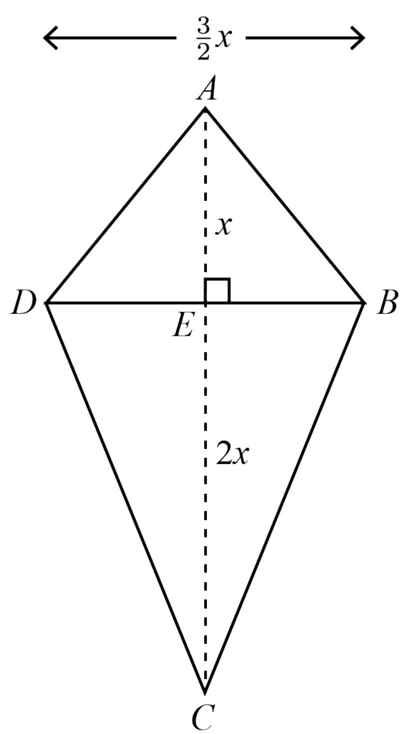
Problem

Xavier made a quilt block to represent an ice cream cone. The quilt block is composed of two isosceles triangles arranged to form a kite. The top triangle represents the ice cream and the bottom triangle represents the cone. The height of the bottom triangle is twice the height of the top triangle. The base of each triangle is $\frac{3}{4}$ of the height of the bottom triangle. If the area of the quilt block of the ice cream cone is 576 units^2, what is its perimeter?

NOTE: You may use the fact that the altitude of an isosceles triangle drawn to the unequal side bisects the unequal side.

Solution

We will call the top triangle $\triangle ABD$ and the bottom triangle $\triangle BCD$, having common base BD. Let E be on BD such that AE is an altitude of $\triangle ABD$. Then, since $\triangle ABD$ is isosceles, $BE = DE$. Notice that CE must also be an altitude of $\triangle BCD$, because if F is the point on BD such that CF is an altitude of $\triangle BCD$, then $BF = DF$. Since both E and F bisect BD, it must be the case that F and E represent the same point. Let $AE = x$. Then $CE = 2x$, and $BD = \left(\frac{3}{4}\right)(2x) = \frac{3}{2}x$.



We can use the area of $ABCD$ to determine the value of x.

$$\begin{aligned}\text{Area of } ABCD &= \text{Area of } \triangle ABD + \text{Area of } \triangle BCD \\ &= \left(\frac{1}{2}\right)(BD)(AE) + \left(\frac{1}{2}\right)(BD)(CE)\end{aligned}$$

Therefore,

$$\begin{aligned}576 &= \left(\frac{1}{2}\right)\left(\frac{3}{2}x\right)(x) + \left(\frac{1}{2}\right)\left(\frac{3}{2}x\right)(2x) \\ &= \frac{3x^2}{4} + \frac{6x^2}{4} \\ &= \frac{9x^2}{4} \\ x^2 &= \frac{576 \times 4}{9} = 256\end{aligned}$$

Thus, $x = 16$, since $x > 0$. Thus, $AE = 16$, $CE = 2x = 32$ and $BD = \frac{3}{2}x = 24$.

Now to find the perimeter of $ABCD$, we need to find the lengths of the sides. Since $\triangle ABD$ is isosceles, it follows that AE bisects BD. Thus, $DE = \frac{1}{2}BD = \frac{24}{2} = 12$. Using the Pythagorean Theorem in $\triangle AED$,

$$\begin{aligned}AD^2 &= AE^2 + DE^2 \\ &= 16^2 + 12^2 \\ &= 400\end{aligned}$$

Thus $AD = 20$, since $AD > 0$. Since $\triangle ABD$ is isosceles, $AB = AD = 20$.

Using the Pythagorean Theorem in $\triangle CED$,

$$\begin{aligned}CD^2 &= CE^2 + DE^2 \\ &= 32^2 + 12^2 \\ &= 1168\end{aligned}$$

Thus $CD = \sqrt{1168}$, since $CD > 0$. Since $\triangle BCD$ is isosceles, $BC = CD = \sqrt{1168}$.

Therefore, the perimeter of $ABCD$ equals $AD + AB + BC + CD = 20 + 20 + \sqrt{1168} + \sqrt{1168} = 40 + 2\sqrt{1168} \approx 108.35$.

Note that we can simplify $\sqrt{1168}$ as follows:

$$\sqrt{1168} = \sqrt{16 \times 73} = 4\sqrt{73}$$

Therefore, the exact perimeter is $40 + 2\sqrt{1168} = 40 + 2(4\sqrt{73}) = 40 + 8\sqrt{73}$.