



Problem of the Week

Problem D and Solution

Slope on a Plane

Problem

Percy drew x and y axes on grid paper and then plotted the point $P(8, 5)$. Quinlan then chose a different point, Q , and said its coordinates were each positive integers less than or equal to 20. Determine the probability that the slope of PQ is 0, 1, or 2.

Solution

First, suppose the slope of PQ is 0. Then Q must have a y -coordinate of 5. Since the coordinates of Q are each positive integers less than or equal to 20, the smallest possible x -coordinate is 1 and the largest is 20. Thus, there are $20 - 1 + 1 = 20$ possible points. However, this includes the point P , so there are $20 - 1 = 19$ possibilities for Q such that PQ has slope 0.

Next, suppose the slope of PQ is 1. Let the coordinates of Q be (a, b) . Then

$$\begin{aligned}\frac{b - 5}{a - 8} &= 1 \\ b - 5 &= a - 8 \\ b &= a - 3\end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(4, 1)$. Similarly, the point with the largest possible value for a is $(20, 17)$. Since $b = a - 3$, there is a possible value for b for each value of a between 4 and 20. Thus, there are $20 - 4 + 1 = 17$ possible points. However, this includes the point P , so there are $17 - 1 = 16$ possibilities for Q such that PQ has slope 1.

Finally, suppose the slope of PQ is 2. Let the coordinates of Q be (a, b) . Then

$$\begin{aligned}\frac{b - 5}{a - 8} &= 2 \\ b - 5 &= 2(a - 8) \\ b - 5 &= 2a - 16 \\ b &= 2a - 11\end{aligned}$$

Since a and b are each positive integers less than or equal to 20, the point with the smallest possible value for a is $(6, 1)$. To determine the point with the largest possible value for a , we first notice that if $a = 20$, then $b > 20$. Then we can set $b = 2a - 11 < 20$. Thus $2a < 31$, or $a < 15.5$. It follows that the point with the largest possible value for a is $(15, 19)$. Since $b = 2a - 11$, there is a possible value for b for each value of a between 6 and 15. Thus, there are $15 - 6 + 1 = 10$ possible points. However, this includes the point P , so there are $10 - 1 = 9$ possibilities for Q such that PQ has slope 2.

Thus, the total number of possibilities for Q such that PQ has slope 0, 1, or 2 is $19 + 16 + 9 = 44$. The total number of possibilities for Q is $20 \times 20 - 1 = 399$. Thus, the probability that the slope of PQ is 0, 1, or 2 is equal to $\frac{44}{399}$, or approximately 11%.