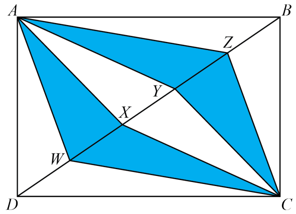

Problem of the Week
Problem D and Solution
They're Blue

Problem

In rectangle $ABCD$, the length of side AB is 7 m and the length of side BC is 5 m. Four points, W, X, Y, and Z, lie on diagonal BD, dividing it into five equal segments. Triangles AWX, AYZ, CWX, and CYZ are then painted blue, as shown. Determine the area of the painted region.

Solution

Solution 1

Using the formula for area of a triangle, area $= \frac{\text{base}\times\text{height}}{2}$, we have area $\triangle ABD = \frac{7\times 5}{2} = \frac{35}{2}$ m^2.

The five triangles $\triangle ADW$, $\triangle AWX$, $\triangle AXY$, $\triangle AYZ$, and $\triangle ABZ$ have the same height, which is equal to the perpendicular distance between BD and A. Since $DW = WX = XY = YZ = ZB$, it follows that the five triangles also have equal bases. Therefore, the area of each of these five triangles is equal to $\frac{1}{5}(\text{area } \triangle ABD) = \frac{1}{5}\left(\frac{35}{2}\right) = \frac{7}{2}$ m^2.

Similarly, the area of $\triangle BCD$ is equal to $\frac{7\times 5}{2} = \frac{35}{2}$ m^2. The five triangles $\triangle CDW$, $\triangle CWX$, $\triangle CXY$, $\triangle CYZ$, and $\triangle CBZ$ also have the same height and equal bases. Therefore, the area of each of these five triangles is equal to $\frac{1}{5}(\text{area } \triangle BCD) = \frac{1}{5}\left(\frac{35}{2}\right) = \frac{7}{2}$ m^2.

Therefore, the area of the painted region is $4\left(\frac{7}{2}\right) = 14$ m^2.

Solution 2

Since $ABCD$ is a rectangle, $\angle DAB = 90^\circ$, so $\triangle ABD$ is a right-angled triangle. We can then use the Pythagorean Theorem to calculate $BD^2 = AB^2 + AD^2 = 7^2 + 5^2 = 49 + 25 = 74$, and so $BD = \sqrt{74}$, since $BD > 0$. Therefore, $DW = WX = XY = YZ = ZB = \frac{1}{5}(BD) = \frac{1}{5}\sqrt{74}$.

Using the formula for area of a triangle, area $= \frac{\text{base}\times\text{height}}{2}$, we have area $\triangle ABD = \frac{7\times 5}{2} = \frac{35}{2}$ m^2.

Let's treat $BD = \sqrt{74}$ as the base of $\triangle ABD$ and let h be the corresponding height. Since the area of $\triangle ABD$ is $\frac{35}{2}$, then we have $\frac{\sqrt{74}\times h}{2} = \frac{35}{2}$ and so $\sqrt{74}\times h = 35$, thus $h = \frac{35}{\sqrt{74}}$.

$\triangle AWX$ and $\triangle AYZ$ both have height $h = \frac{35}{\sqrt{74}}$ and base $\frac{\sqrt{74}}{5}$, so area $\triangle AWX = \text{area } \triangle AYZ = \frac{1}{2}\left(\frac{\sqrt{74}}{5}\right)\left(\frac{35}{\sqrt{74}}\right) = \frac{7}{2}$ m^2.

Similarly, $\triangle CWX$ and $\triangle CYZ$ both have height $h = \frac{35}{\sqrt{74}}$ and base $\frac{\sqrt{74}}{5}$, so area $\triangle CWX = \text{area } \triangle CYZ = \frac{1}{2}\left(\frac{\sqrt{74}}{5}\right)\left(\frac{35}{\sqrt{74}}\right) = \frac{7}{2}$ m^2.

Therefore, the area of the painted region is $4\left(\frac{7}{2}\right) = 14$ m^2.