The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING

Problem of the Week Problems and Solutions 2022 - 2023

Problem C (Grade 7/8)

Themes

(Click on a theme name to jump to that section.)

Number Sense (N)

Geometry & Measurement (G)

Algebra (A)

Data Management (D)

Computational Thinking (C)

The problems in this booklet are organized into themes.
A problem often appears in multiple themes.

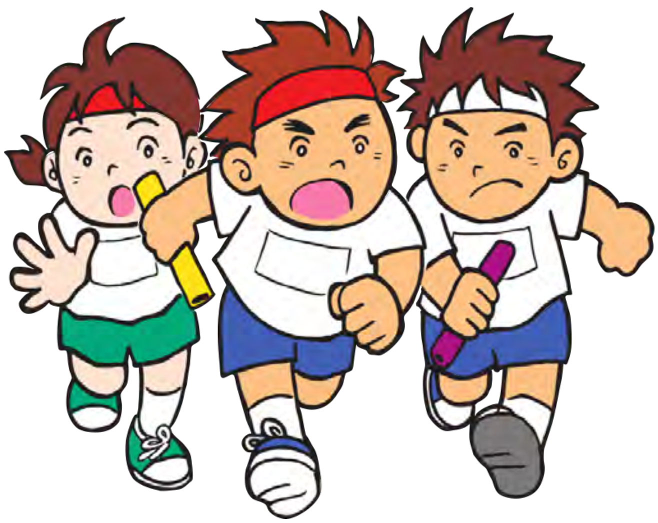
Number Sense (N)

TAKE ME TO THE COVER

Problem of the Week
Problem C
Wacky Race

Four students participate as a team in a 1000 m wacky relay race. In a wacky relay race, the students each run a portion of the 1000 m length, but they do not run equal lengths. Andrea and Billie run $\frac{1}{8}$ and $\frac{1}{5}$ of the total length, respectively. Carol runs the average of what Andrea and Billie run. Dana runs the remainder of the length.

Determine the fraction of the total length that Dana runs.



Problem of the Week
Problem C and Solution
Wacky Race

Problem

Four students participate as a team in a 1000 m wacky relay race. In a wacky relay race, the students each run a portion of the 1000 m length, but they do not run equal lengths. Andrea and Billie run $\frac{1}{8}$ and $\frac{1}{5}$ of the total length, respectively. Carol runs the average of what Andrea and Billie run. Dana runs the remainder of the length.

Determine the fraction of the total length that Dana runs.

Solution

In the first solution, we solve the problem by working with the fractions and without calculating the length that each person runs. In the second solution, we determine the fraction of the length that Dana runs by calculating the distance Dana runs and dividing by the total length of the race.

Solution 1

Carol runs the average of $\frac{1}{8}$ and $\frac{1}{5}$ of the total length of the race.

Therefore, Carol runs $\dfrac{\frac{1}{8}+\frac{1}{5}}{2} = \dfrac{\frac{5}{40}+\frac{8}{40}}{2} = \dfrac{\frac{13}{40}}{2} = \dfrac{13}{80}$ of the race.

Dana runs the remainder of the race.

Therefore, Dana runs $1 - \frac{1}{8} - \frac{1}{5} - \frac{13}{80} = \frac{80}{80} - \frac{10}{80} - \frac{16}{80} - \frac{13}{80} = \frac{41}{80}$ of the race. Dana runs just over half of the race.

Solution 2

Andrea runs $\frac{1}{8}$ of the race, so she runs $\frac{1}{8} \times 1000 = 125$ m.

Billie runs $\frac{1}{5}$ of the race, so they run $\frac{1}{5} \times 1000 = 200$ m.

Carol runs the average of what Andrea and Billie run.

Therefore, Carol runs $\frac{125+200}{2} = \frac{325}{2} = 162.5$ m.

Dana runs the remainder of the race.

Therefore, Dana runs $1000 - 125 - 200 - 162.50 = 512.5$ m.

That is, Dana runs $\frac{512.5}{1000} = \frac{5125}{10000} = \frac{41}{80}$ of the race.

Problem of the Week

Problem C

Two Squares

Simone has a rope that is 60 cm long. They cut the rope into two pieces so that the ratio of the lengths of the two pieces is $7:3$. Each piece of the rope is then arranged, with its two ends touching, to form a square.

What is the total area of the two squares?

Themes Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Two Squares

Problem

Simone has a rope that is 60 cm long. They cut the rope into two pieces so that the ratio of the lengths of the two pieces is $7:3$. Each piece of the rope is then arranged, with its two ends touching, to form a square.

What is the total area of the two squares?

Solution

Since the rope is cut in the ratio of $7:3$, the ratio of the longer piece to the whole rope will be $7:(7+3)$ or $7:10$. This means the length of the longer piece will be $\frac{7}{10}$ of the length of the whole rope. Similarly, the length of the shorter piece would be $\frac{3}{10}$ of the length of the whole rope. Therefore, the longer piece is $\frac{7}{10}$ of 60 or $\frac{7}{10}\times 60 = 42$ cm long. Also, the shorter piece is $\frac{3}{10}$ of 60 or $\frac{3}{10}\times 60 = 18$ cm long.

Each of the two pieces is then used to form a square. The perimeter of each square is the length of the rope used to form it. The side length of the longer square is $42 \div 4 = 10.5$ cm and the side length of the shorter square is $18 \div 4 = 4.5$ cm.

To find the area of each square, we multiply the length by the width. In effect, to find the area of the square, we square the side length. Thus, the area of the larger square is $10.5\times 10.5 = 10.5^2 = 110.25\text{ cm}^2$ and the area of the smaller square is $4.5\times 4.5 = 4.5^2 = 20.25\text{ cm}^2$.

Therefore, the total area of the two squares is $110.25 + 20.25 = 130.5\text{ cm}^2$.

For Further Thought:

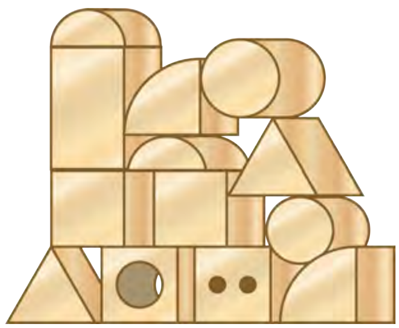
The ratio of the area of the larger square to the area of the smaller square is

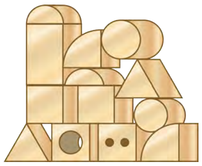
$$110.25 : 20.25 = 11\,025 : 2025 = 441 : 81 = 49 : 9 = 7^2 : 3^2$$

Notice that the ratio of the perimeter of the larger square to the perimeter of the smaller square is $7:3$ and the ratio of their areas is $7^2:3^2$. In general, if the ratio of the perimeters of two squares is $a:b$, is it true that the ratio of the areas of the two squares is $a^2:b^2$?

Problem of the Week
Problem C
Playing with Blocks

Agnes, Evangelina, Isabela, Omar, and Yuri each made a tower using wooden blocks. Each person used a different number of blocks in their tower, and the mean (average) number of blocks in each tower was 25. Yuri used the most blocks in her tower, and Agnes used the fewest blocks in her tower. If Yuri used 32 blocks, determine the minimum possible number of blocks that Agnes could have used.





Problem of the Week
Problem C and Solution
Playing with Blocks

Problem

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Solution

To calculate the mean (average) of a set of values, we first calculate the sum of the values in the set, and then divide that by the number of values in the set. It follows that the sum of the values in the set is equal to their average multiplied by the number of values in the set.

Since the average number of blocks in each tower was 25, and there were 5 towers, it follows that the total number of blocks used was $25 \times 5 = 125$. Yuri's tower used 32 blocks, so the remaining towers used a total of $125 - 32 = 93$ blocks.

To find the minimum possible number of blocks in Agnes' tower, we let the other three towers use the greatest possible number of blocks. We know Yuri's tower used the most blocks, and each tower used a different number of blocks. So the other three towers could have used at most $31,\ 30$, and 29 blocks, in some order.

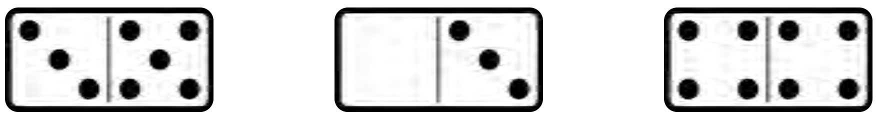
The minimum possible number of blocks that Agnes could have used is therefore $93 - 31 - 30 - 29 = 3$.

As a side note, if each person could have used the same number of blocks, then the minimum possible number of blocks that Agnes could have used would have been $93 - 32 - 32 - 32 = -3$. However it's not possible to use a negative number of blocks, so Agnes must have used at least 1 block. There would be a few different options for the number of blocks in each tower in order to make this possible. For example, the towers could contain $1,\ 28,\ 32,\ 32$, and 32 blocks each, or $1,\ 30,\ 30,\ 32$, and 32 blocks each.

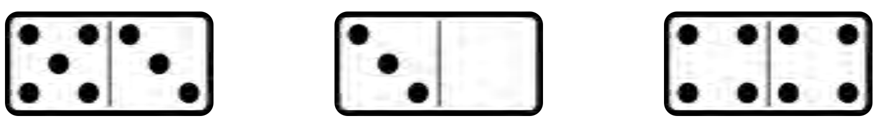
Problem of the Week
Problem C
Domi Knows

A domino tile is a rectangular tile with a line dividing its face into two square ends. Each end is marked with a number of dots (also called pips) or is blank.

The first domino shown below is a $[3, 5]$ domino, since there are 3 pips on its left end and 5 pips on its right end. The second domino shown below is a $[0, 3]$ domino, since there are 0 pips on its left end and 3 pips on its right end. The third domino shown below is a $[4, 4]$ domino, since there are 4 pips on its left end and 4 pips on its right end.



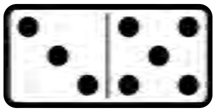
We can also rotate the domino tiles. The first domino shown below is a $[5, 3]$ domino, since there are 5 pips on its left end and 3 pips on its right end. However, since this tile can be obtained by rotating the $[3, 5]$ tile, $[5, 3]$ and $[3, 5]$ represent the same domino. Similarly, the second domino shown below is a $[3, 0]$ domino. Again, note that $[3, 0]$ and $[0, 3]$ represent the same domino.



A 2-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 2, with no two dominoes being the same. A 2-set of dominoes has the following six tiles: $[0, 0]$, $[0, 1]$, $[0, 2]$, $[1, 1]$, $[1, 2]$, $[2, 2]$. Notice that the three dominoes $[1, 0]$, $[2, 0]$, and $[2, 1]$ are not listed because they are the same as the three dominoes $[0, 1]$, $[0, 2]$, and $[1, 2]$.

Similarly, a 12-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 12, with no two dominoes being the same.

Domi purchased a 12-set of dominoes. How many tiles are in the set?



Problem of the Week
Problem C and Solution
Domi Knows

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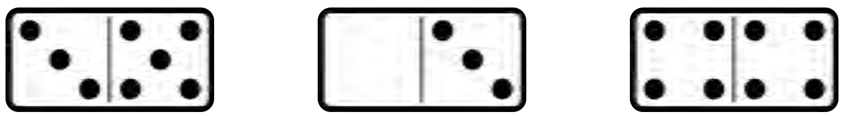

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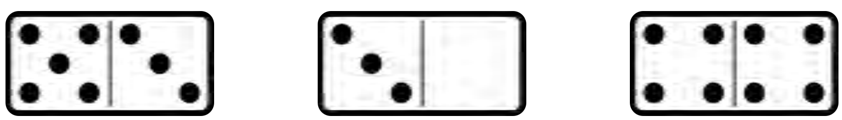

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Similarly, a 12-set of dominoes contains all possible tiles with the number of pips on any end ranging from 0 to 12, with no two dominoes being the same.

Domi purchased a 12-set of dominoes. How many tiles are in the set?

Solution

Since rotating a domino tile does not change the domino, we will orient each domino so that the smaller number is always on the left end of the domino. Then, for each possible number on the left end of the domino, we will examine the possible numbers that can occur on the right end of the domino, and thus how many dominoes in the set have that number on the left end. We compile this information in a table.

Number on Left End of Domino	Possible Numbers on Right End of Domino	Total Number of Dominoes
0	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	13
1	1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	12
2	2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12	11
3	3, 4, 5, 6, 7, 8, 9, 10, 11, 12	10
4	4, 5, 6, 7, 8, 9, 10, 11, 12	9
5	5, 6, 7, 8, 9, 10, 11, 12	8
6	6, 7, 8, 9, 10, 11, 12	7
7	7, 8, 9, 10, 11, 12	6
8	8, 9, 10, 11, 12	5
9	9, 10, 11, 12	4
10	10, 11, 12	3
11	11, 12	2
12	12	1

Therefore, the total number of dominoes in a 12-set is

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = 91$$

Did you know?

A quick way to calculate the sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13$$

is as

$$\frac{(13)(13 + 1)}{2}$$

That is, $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 12 + 13 = \frac{(13)(13 + 1)}{2}$.

Can you convince yourself that this is true?

In general, it can be shown that if n is a positive integer, then the sum of the integers from 1 to n is equal to $\frac{n \times (n + 1)}{2}$.

In other words,

$$1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}$$

Problem of the Week

Problem C

Gone Shopping

While grocery shopping, Terry has a way to approximate the total cost of his purchases. He simply approximates that each item will cost \$3.00.

One day, Terry purchased 20 items. He purchased items that each had an actual cost of either \$1.00, \$3.00, or \$7.50. Exactly seven of the purchased items had an actual cost of \$3.00. If the total actual cost of the 20 items was the same as the total approximated cost, how many items had an actual cost of \$7.50?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Gone Shopping

Problem

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One day, Terry purchased 20 items. He purchased items that each had an actual cost of either \$1.00, \$3.00, or \$7.50. Exactly seven of the purchased items had an actual cost of \$3.00. If the total actual cost of the 20 items was the same as the total approximated cost, how many items had an actual cost of \$7.50?

Solution

The total approximated cost for the 20 items is $20 \times \$3 = \$60$. Since the total actual cost is the same as the total approximated cost, the total actual cost for the 20 items is \$60. Since 7 of the items cost \$3.00, it cost Terry $7 \times \$3 = \$21$ to buy these items. Therefore, the remaining $20 - 7 = 13$ items cost $\$60 - \$21 = \$39$.

From this point, we will continue with two different solutions.

Solution 1

In this solution, we will use systematic trial-and-error to solve the problem.

Let s represent the number of items Terry bought with an actual cost of \$7.50 and d represent the number of items that Terry bought with an actual cost \$1.00. Then the total cost of the \$7.50 items would be $7.5s$. Also, the total cost of the \$1.00 items would be $1d = d$. Since Terry's total remaining cost was \$39, then $7.5s + d = 39$. We also know that $s + d = 13$.

At this point we can systematically pick values for s and d that add to 13 and substitute into the equation $7.5s + d = 39$ to find the combination that works. (We can observe that $s < 6$ since $7.5 \times 6 = 45 > 39$. If this were the case, then d would have to be a negative number.)

Let's start with $s = 3$. Then $d = 13 - 3 = 10$. The cost of these items would be $7.5 \times 3 + 10 = 22.50 + 10 = \$32.50$, which is less than \$39.

So let's try $s = 4$. Then $d = 13 - 4 = 9$. The cost of these items would be $7.5 \times 4 + 9 = 30 + 9 = \$39$, which is the amount we want.

Therefore, Terry purchased 4 items that cost \$7.50.

Solution 2

In this solution, we will use algebra to solve the problem.

Let s represent the number of items that cost \$7.50. Therefore, $(13 - s)$ represents the number of items that cost \$1.00. Also, the total cost of the \$7.50 items would be $7.5s$, the total cost of the \$1.00 items would be $1 \times (13 - s) = 13 - s$, and the total of these two is $7.5s + 13 - s = 6.5s + 13$.

Since Terry's total remaining cost was \$39.00, we must have

$$\begin{aligned} 6.5s + 13 &= 39 \\ 6.5s + 13 - 13 &= 39 - 13 \\ 6.5s &= 26 \\ \frac{6.5s}{6.5} &= \frac{26}{6.5} \\ s &= 4 \end{aligned}$$

Therefore, Terry purchased 4 items that cost \$7.50.

Problem of the Week
Problem C
Teacher Road Trip 1

To help pass time on a long bus ride, 35 math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first teacher said the number 2, the second teacher said the number 8, and every teacher after that said the sum of the two previous terms. Thus,

- the third teacher said the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher said the sum of the second and third terms, which is $8 + 10 = 18$.

Once the final teacher said their number, the 25^{th} teacher announced they had made a mistake and their number should have been one more than what they had said. How much larger should the final teacher's number have been?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Teacher Road Trip 1

Problem

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Once the final teacher said their number, the 25^{th} teacher announced they had made a mistake and their number should have been one more than what they had said. How much larger should the final teacher's number have been?

Solution

Solution 1

We will write out the sequence of numbers the teachers actually said, and then the sequence of numbers they should have said, and then find the difference between the last term in each sequence.

Here are the first 24 numbers that the teachers said:

2, 8, 10, 18, 28, 46, 74, 120, 194, 314, 508, 822, 1330, 2152, 3482, 5634, 9116, 14 750, 23 866, 38 616, 62 482, 101 098, 163 580, 264 678

Here are the correct 25^{th} to 35^{th} numbers that the teachers should have said:

428 258, 692 936, 1 121 194, 1 814 130, 2 935 324, 4 749 454, 7 684 778, 12 434 232, 20 119 010, 32 553 242, 52 672 252

Here are the 25^{th} to 35^{th} numbers that the teachers actually said:

428 257, 692 935, 1 121 192, 1 814 127, 2 935 319, 4 749 446, 7 684 765, 12 434 211, 20 118 976, 32 553 187, 52 672 163

The difference between the correct and incorrect 35^{th} number is $52\,672\,252 - 52\,672\,163 = 89$. Therefore, the 35^{th} number was off by 89, and so the final teacher's number should have been 89 larger than the number they had said.

Solution 2

In this solution we will solve the problem without actually calculating all the terms in the sequence.

We know the 25$^{\text{th}}$ term is off by 1. Therefore, the next terms will be as follows.

- The 26$^{\text{th}}$ term will also be off by 1 since it equals the sum of the 24$^{\text{th}}$ term (which is unchanged) and the 25$^{\text{th}}$ term (which is off by 1).
- The 27$^{\text{th}}$ term will be off by 2 since it is the sum of the 25$^{\text{th}}$ term (which is off by 1) and the 26$^{\text{th}}$ term (which is off by 1).
- The 28$^{\text{th}}$ term will be off by 3 since it is the sum of the 26$^{\text{th}}$ term (which is off by 1) and the 27$^{\text{th}}$ term (which is off by 2).

This pattern will continue on, so we can summarize it in a table.

Term Number	Amount Below the Correct Value
24	0
25	1
26	1
27	2
28	3
29	5
30	8
31	13
32	21
33	34
34	55
35	89

Therefore, the 35$^{\text{th}}$ term was off by 89, and so the final teacher's number should have been 89 larger than the number they had said.

Notice that the terms in the right column of the table follow the same rule as the original question. That is, each term is the sum of the previous two terms.

For Further Thought: The last 11 numbers in the right column of the table are the first 11 numbers of a famous sequence known as the Fibonacci Sequence. You may wish to investigate the Fibonacci Sequence further.

Problem of the Week

Problem C

Divisors and Number

Your friend Cael always likes challenging you. One challenge is called "*Divisors and Number*". Cael will tell you certain facts about the divisors of a number and then challenge you to find the number. Here is Cael's challenge.

"I am looking for a positive integer with exactly eight positive divisors, two of which are 21 and 33."

Determine Cael's number.

Problem of the Week
Problem C and Solution
Divisors and Number

Problem

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"I am looking for a positive integer with exactly eight positive divisors, two of which are 21 and 33."

Determine Cael's number.

Solution

Let n represent the number we are looking for.

We know that four of the positive divisors of n are 1, 21, 33, and n. In our solution we will first find the remaining four positive divisors and then determine n.

Since 21 is a divisor of n and $21 = 3 \times 7$, then 3 and 7 must also be divisors of n.

Since 33 is a divisor of n and $33 = 3 \times 11$, then 11 must also be a divisor of n.

Since 7 is a divisor of n and 11 is a divisor of n, and since 7 and 11 have no common divisors, then $7 \times 11 = 77$ must also be a divisor of n.

We have found all eight of the positive divisors of the unknown number. The positive divisors are 1, 3, 7, 11, 21, 33, 77, and n. We now need to determine n.

From the list of divisors, we can see that the prime factors of n are 3, 7, and 11. It follows that $n = 3 \times 7 \times 11 = 231$.

Therefore, Cael's number is 231.

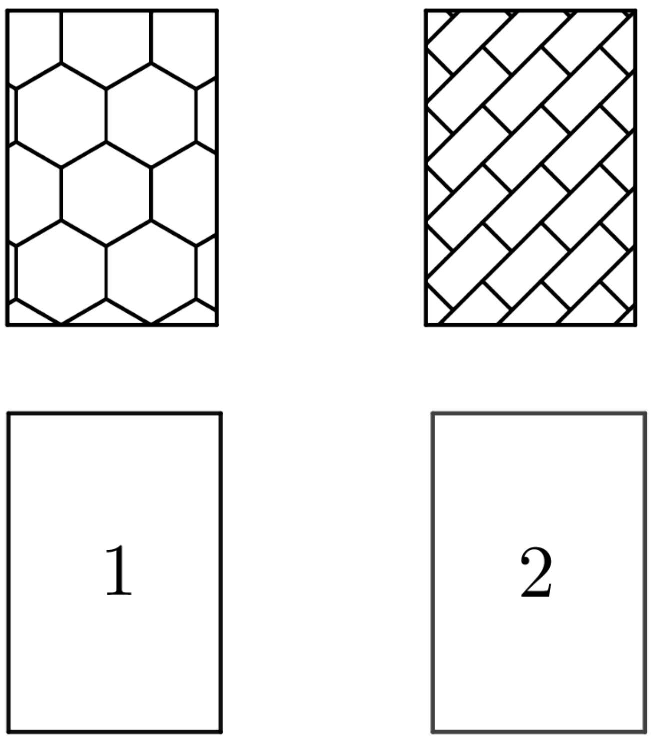
Problem of the Week

Problem C

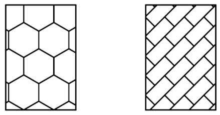
Two Bricks

Dhvanil has a large number of cards. On the back of each card there is either a honeycomb pattern (hexagons) or a brick pattern (rectangles). On the front of each card there is either a 1 or a 2. As Dhvanil went through all the cards, he found that 30% of the cards have a honeycomb pattern on the back. Of the cards with a brick pattern on the back, 80% have a 1 on the front.

Determine the percentage of all the cards that have a brick pattern on the back and a 2 on the front.



Theme Number Sense



Problem of the Week

Problem C and Solution

Two Bricks

Problem

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Determine the percentage of all the cards that have a brick pattern on the back and a 2 on the front.

Solution

Solution 1

Let's suppose that Dhvanil has 100 cards.

If 30% of the cards have a honeycomb pattern on the back, that means that $0.3 \times 100 = 30$ cards have a honeycomb pattern on the back. Therefore, $100 - 30 = 70$ cards have a brick pattern on the back.

Of the cards with a brick pattern on the back, 80% have a 1 on the front. Therefore, there are $0.8 \times 70 = 56$ cards with a brick pattern on the back and 1 on the front. Therefore, $70 - 56 = 14$ cards have a brick pattern on the back and a 2 on the front.

Therefore, the percentage of all cards with a brick pattern on the back and a 2 on the front is $\dfrac{14}{100} \times 100\% = 14\%$.

Solution 2

If 30% of the cards have a honeycomb pattern on the back, that means that 70% of the cards have a brick pattern on the back.

Of the cards with a brick pattern on the back, 80% have a 1 on the front. Therefore, of the cards with a brick pattern on the back, 20% have a 2 on the front. Since $0.2 \times 0.70 = 0.14$, 14% of all of the cards have a brick pattern on the back and a 2 on the front.

Problem of the Week
Problem C
What's the Score?

In gym class, the yellow team and the blue team played soccer. Ali doesn't remember the final score of the game, but she does remember the following.

- There were six goals scored in total.
- Neither team scored more than two goals in a row at any point in the game.
- The blue team won the game.

Determine all the possible final scores and the different ways each score could have been obtained.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
What's the Score?

Problem

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- There were six goals scored in total.
- Neither team scored more than two goals in a row at any point in the game.
- The blue team won the game.

Determine all the possible final scores and the different ways each score could have been obtained.

Solution

In order to win, the blue team must have scored more goals than the yellow team. Since there were six goals scored in total, the only possibilities for the final scores are $4 - 2,\ 5 - 1$, or $6 - 0$ for the blue team.

Next we need to check which of these scores are possible, given that neither team scored more than two goals in a row at any point in the game.

- Is a final score of $6 - 0$ possible?

  We can easily eliminate $6 - 0$, since the blue team would have had to score more than two goals in a row.

- Is a final score of $5 - 1$ possible?

  This would mean that the blue team scored 5 goals and the yellow team scored 1 goal. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team. These are all shown below.

  $YBBBBB$, $BYBBBB$, $BBYBBB$, $BBBYBB$, $BBBBYB$, $BBBBBY$

  As we can see, in all of these arrangements, the blue team scored more than two goals in a row. Thus, a final score of $5 - 1$ is not possible.

- Is a final score of $4-2$ possible?

  This would mean that the blue team scored 4 goals and the yellow team scored 2 goals. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team.

  – Case 1: The yellow team scored their 2 goals in a row. The possible arrangements are shown below.

  $$YYBBBB,\ BYYBBB,\ BBYYBB,\ BBBYYB,\ BBBBYY$$

  In this case, there is only 1 arrangement where neither team scored more than two goals in a row, namely $BBYYBB$.

  – Case 2: The yellow team did not score their 2 goals in a row. The possible arrangements are shown below.

  $$YBYBBB,\ YBBYBB,\ YBBBYB,\ YBBBBY,\ BYBYBB,$$
  $$BYBBYB,\ BYBBBY,\ BBYBYB,\ BBYBBY,\ BBBYBY$$

  In this case, there are 5 arrangements where neither team scored more than two goals in a row, namely
  $YBBYBB$, $BYBYBB$, $BYBBYB$, $BBYBYB$, and $BBYBBY$.

  Therefore, the only possible final score is $4-2$ for the blue team, and it could be obtained in the following six ways.

  $BBYYBB$, $YBBYBB$, $BYBYBB$, $BYBBYB$, $BBYBYB$, $BBYBBY$

Problem of the Week
Problem C
Again and Again

The fraction $\frac{1}{7}$ is equal to the repeating decimal $0.\overline{142857}$.

Which digit occurs in the 2023^{rd} place after the decimal point?

0.142857142857142...

Problem of the Week
Problem C and Solution
Again and Again

Problem

The fraction $\frac{1}{7}$ is equal to the repeating decimal $0.\overline{142857}$.

Which digit occurs in the 2023^{rd} place after the decimal point?

Solution

The digits after the decimal point occur in repeating blocks of the 6 digits 142857.

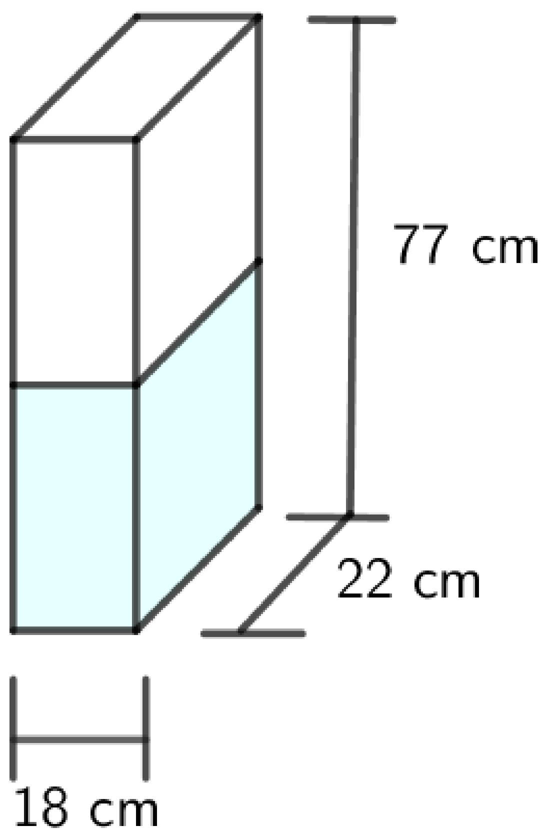
Since $\frac{2023}{6} = 337.1\overline{6} = 337\frac{1}{6}$, it follows that the 2023^{rd} digit after the decimal point occurs after 337 complete repeating blocks of the 6 digits.

In 337 complete repeating blocks, there are $337 \times 6 = 2022$ digits in total. The 2023^{rd} digit is then the next digit. This corresponds to the first digit in the repeating block, which is 1.

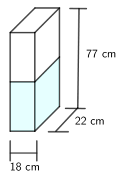
Therefore, the digit 1 occurs in the 2023^{rd} place after the decimal point.

Problem of the Week
Problem C
Ice Box

A metal box in the form of a rectangular prism has an 18 cm by 22 cm base and a height of 77 cm. The box is to be filled with water, which will then be frozen. When water freezes it expands by approximately 10%. Determine the maximum depth to which the box can be filled with water so that when the water freezes, the ice does not go above the top of the container.



Themes Geometry & Measurement, Number Sense



Problem of the Week

Problem C and Solution

Ice Box

Problem

A metal box in the form of a rectangular prism has an 18 cm by 22 cm base and a height of 77 cm. The box is to be filled with water, which will then be frozen. When water freezes it expands by approximately 10%. Determine the maximum depth to which the box can be filled with water so that when the water freezes, the ice does not go above the top of the container.

Solution

Solution 1

To determine the volume of a rectangular prism, we multiply its length, width, and height together. So, the maximum volume of the metal box is

$$18 \times 22 \times 77 = 30\,492 \text{ cm}^3$$

Let the original depth of water in the metal box be h cm.

The water volume before freezing is $18 \times 22 \times h = (396 \times h)$ cm^3. After the water freezes, the volume increases by 10% to 110% of its current volume. That is, after freezing the volume is

$$110\% \text{ of } 396 \times h = 1.1 \times 396 \times h = (435.6 \times h) \text{ cm}^3$$

But the volume after freezing is the maximum volume, $30\,492$ cm^3. Therefore, $435.6 \times h = 30\,492$ and it follows that $h = 30\,492 \div 435.6 = 70$ cm.

Therefore, the maximum depth to which the box can be filled is 70 cm.

Solution 2

In this solution we note that the length and width remain the same in the volume calculations before and after the water freezes. We need only concern ourselves with the change in the depth of the water.

Let the original depth of water in the container be h cm.

After freezing, the depth increases by 10% to 110% of its depth before freezing. So, after freezing the depth will be 110% of $h = 1.1 \times h = 77$ cm, the maximum height of the container. Then $h = 77 \div 1.1 = 70$ cm.

Therefore, the maximum depth to which the box can be filled is 70 cm.

Problem of the Week

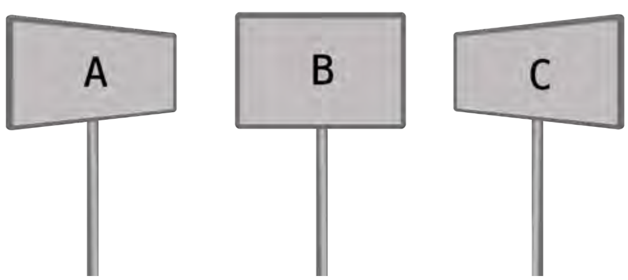
Problem C

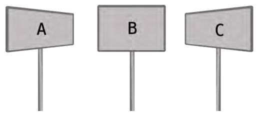
Go Back

Gwen has been given the ability to time travel by walking along three different trails. She can walk on any trail as often as she wishes, but can only walk on one trail at a time. She must walk on the trails using the following rules.

- When she walks on Trail A, she must take 7 steps forward. This will allow her to travel 4 months backward in time.
- When she walks on Trail B, she must take 5 steps backward. This will allow her to travel 7 months backward in time.
- When she walks on Trail C, she must take 2 steps forward. This will allow her to travel 3 months backward in time.

One day she travels 5 years into the past. She made a total of 25 steps backward and walked on the three trails a total of 12 times. How many steps forward did she take?





Problem of the Week

Problem C and Solution

Go Back

Problem

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- When she walks on Trail C, she must take 2 steps forward. This will allow her to travel 3 months backward in time.

One day she travels 5 years into the past. She made a total of 25 steps backward and walked on the three trails a total of 12 times. How many steps forward did she take?

Solution

Gwen travelled 5 years back in time, which is equivalent to travelling $5 \times 12 = 60$ months back in time.

Trail B is the only trail that requires that she step backward. For every 5 steps backward, she travels 7 months back in time. Therefore, for 25 steps backward, she used Trail B $25 \div 5 = 5$ times and travelled back in time $5 \times 7 = 35$ months.

She still needs to travel $60 - 35 = 25$ more months back in time. She has used Trail B 5 times, and since she uses the trails a total of 12 times, she has $12 - 5 = 7$ trail uses left. She can now only use Trail A and Trail C. We will present two solutions from this point.

Solution 1

If Gwen uses Trail A and Trail C one time each, she travels a total of 7 months back in time. If she uses Trail A and Trail C three times each, this accounts for six uses and she travels a total of $7 \times 3 = 21$ months back in time. She has one use left and still needs to travel 4 more months back in time. This can be accomplished by using Trail A once more.

It follows that Trail A is used 4 times and Trail C is used 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

Note that we could also have looked at each of the possibilities for using Trail A. Since there are a total of 7 trail uses for Trails A and C, the minimum number of uses for Trail A would be 0 and the maximum number of uses for Trail A would be 7. Once the number of uses for Trail A is selected, the number of uses for Trail C can be determined by subtracting the number of uses for Trail A from 7. For each combination we could determine the number of months travelled back in time. Once the correct combination is determined the total number of forward steps can be calculated. This is summarized in a table.

Uses of Trail A	Uses of Trail C	Months Travelled Back in Time
0	7	$0 \times 4 + 7 \times 3 = 0 + 21 = 21$
1	6	$1 \times 4 + 6 \times 3 = 4 + 18 = 22$
2	5	$2 \times 4 + 5 \times 3 = 8 + 15 = 23$
3	4	$3 \times 4 + 4 \times 3 = 12 + 12 = 24$
4	3	$4 \times 4 + 3 \times 3 = 16 + 9 = 25$
5	2	$5 \times 4 + 2 \times 3 = 20 + 6 = 26$
6	1	$6 \times 4 + 1 \times 3 = 24 + 3 = 27$
7	0	$7 \times 4 + 0 \times 3 = 28 + 0 = 28$

Only one combination gives the correct number of trail uses and the correct number of months travelled back in time. Using only Trail A and Trail C a total of 7 times, if we want to travel back in time 25 months we need to use Trail A 4 times and Trail C 3 times. The total number of forward steps is $4 \times 7 + 3 \times 2 = 28 + 6 = 34$.

Solution 2

This solution is presented for you to get a glimpse of what is coming in future mathematics courses.

Let a be the number of uses of Trail A, b be the number of uses of Trail B, and c be the number of uses of Trail C. Since the total number of uses is 12, then $a + b + c = 12$.

The total number of backward steps is 25 and Trail B is the only trail requiring backward steps. Since each use of Trail B requires 5 backward steps, then we require a total of 5 uses of Trail B to go back 25 steps. It follows that $b = 5$ and the equation $a + b + c = 12$ becomes $a + 5 + c = 12$, which simplifies to $a + c = 7$.

In using Trail B 5 times, Gwen travels a total of $5 \times 7 = 35$ months back in time. She needs to travel a total of 5 years or 60 months back in time. Thus, using Trail A and Trail C, she needs to travel $60 - 35 = 25$ more months back in time. Since she travels 4 months backward with each use of Trail A and 3 months backward with each use of Trail C, we need $4a + 3c = 25$.

Rearranging the equation $a + c = 7$, we obtain $c = 7 - a$. We can substitute for c in the equation $4a + 3c = 25$.

$$\begin{aligned} 4a + 3c &= 25 \\ 4a + 3(7 - a) &= 25 \\ 4a + 21 - 3a &= 25 \\ a + 21 &= 25 \\ a &= 4 \end{aligned}$$

We can substitute $a = 4$ into the equation $a + c = 7$ to determine that $c = 3$.

For each use of Trail A, 7 forward steps are required. Therefore, Gwen steps forward $7a$ steps using Trail A. For each use of Trail C, 2 forward steps are required. Therefore, Gwen steps forward $2c$ steps using Trail C. The total number of steps forward is $7a + 2c$. Since $a = 4$ and $c = 3$, the total number of forward steps is $7(4) + 2(3) = 28 + 6 = 34$.

Problem of the Week

Problem C

Missing the Fives I

Bobbi lists the positive integers, in order, excluding all multiples of 5. Her resulting list is

$$1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, \ldots$$

How many integers has Bobbi listed just before she leaves out the 2023rd multiple of 5?

Problem of the Week
Problem C and Solution
Missing the Fives I

Problem
Bobbi lists the positive integers, in order, excluding all multiples of 5. Her resulting list is

$$1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, \ldots$$

How many integers has Bobbi listed just before she leaves out the 2023rd multiple of 5?

Solution

Solution 1

In the list of integers beginning at 1, the 2023th multiple of 5 is $2023 \times 5 = 10\,115$. Thus, Bobbi has listed each of the integers from 1 to 10 114 with the exception of the positive multiples of 5 less than 10 115. Since 10 115 is the 2023rd multiple of 5, Bobbi will not write 2022 multiples of 5.

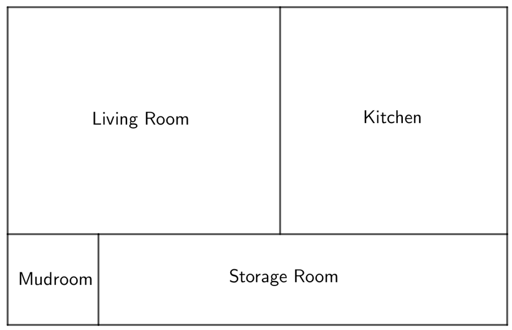
Therefore, just before Bobbi leaves out the 2023rd multiple of 5, she has listed $10\,114 - 2022 = 8092$ integers.

Solution 2

Beginning at 1, each group of five integers has one integer that is a multiple of 5. For example, the first group of five integers, 1, 2, 3, 4, 5, has one multiple of 5 (namely 5), and the second group of five integers, 6, 7, 8, 9, 10, has one multiple of 5 (namely 10). In Bobbi's list of integers, she leaves out the integers that are multiples of 5, and so in every group of five integers, Bobbi lists four of these integers. Thus, just before Bobbi leaves out the 2023rd multiple of 5, there were 2023 of these groups. Therefore, she has listed $2023 \times 4 = 8092$ integers.

Problem of the Week
Problem C
Dollhouse

The first level of a dollhouse is in the shape of a rectangle. Its floor plan is shown in the following diagram.



Both the mudroom and the kitchen are square with areas of 400 cm^2 and 2500 cm^2, respectively. The living room is rectangular with an area of 3000 cm^2.

Determine the area of the rectangular storage room.

Themes Geometry & Measurement, Number Sense

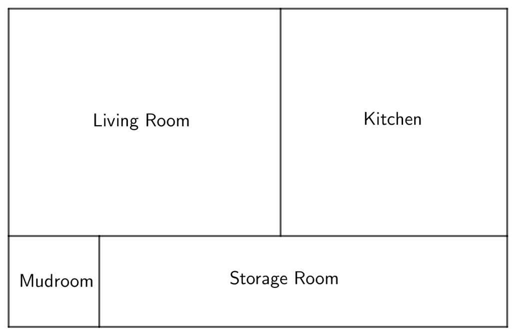
Problem of the Week

Problem C and Solution

Dollhouse

Problem

The first level of a dollhouse is in the shape of a rectangle. Its floor plan is shown in the following diagram.



Both the mudroom and the kitchen are square with areas of 400 cm^2 and 2500 cm^2, respectively. The living room is rectangular with an area of 3000 cm^2.

Determine the area of the rectangular storage room.

Solution

Let the width of a room be the vertical length of the room on the diagram. Let the length of a room be the horizontal length of the room on the diagram.

The kitchen is a square and has an area of 2500 cm^2. Its length and width must both be 50 cm since $50 \times 50 = 2500$ cm^2. The living room and kitchen have the same width. So the width of the living room must also be 50 cm. But the area of the living room is 3000 cm^2, so the length of the living room is 60 cm since $50 \times 60 = 3000$ cm^2.

The mudroom is a square and has an area of 400 cm^2. Its length and width must both be 20 cm since $20 \times 20 = 400$ cm^2. The mudroom and storage room have the same width. So the width of the storage room must also be 20 cm.

Now the length of the whole house can be calculated in two ways. We will equate these two expressions to find the length of the storage room.

$$\begin{aligned} \text{mudroom length} + \text{storage room length} &= \text{living room length} + \text{kitchen length} \\ 20 + \text{storage room length} &= 60 + 50 \\ 20 + \text{storage room length} &= 110 \\ \text{storage room length} &= 90 \text{ cm} \end{aligned}$$

Since the width of the storage room is 20 cm and the length of the storage room is 90 cm, the area of the storage room is $20 \times 90 = 1800$ cm^2.

Problem of the Week

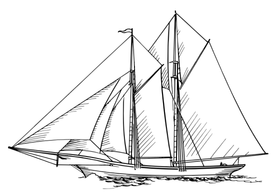
Problem C

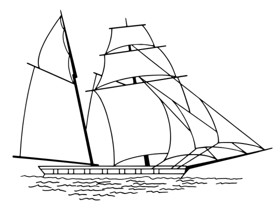
See You No More

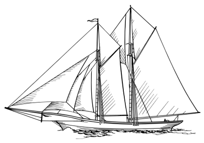
Two boats are travelling away from each other in opposite directions. One boat is travelling east at the constant speed of 8 km/h and the other boat is travelling west at a different constant speed.

At one point, the boat travelling east was 200 m east of the boat travelling west, but 15 minutes later they lose sight of each other.

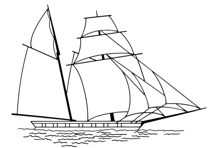
If the visibility at sea that day was 5 km, determine the constant speed of the boat travelling west.







Problem of the Week
Problem C and Solution
See You No More



Problem

Two boats are travelling away from each other in opposite directions. One boat is travelling east at the constant speed of 8 km/h and the other boat is travelling west at a different constant speed.

At one point, the boat travelling east was 200 m east of the boat travelling west, but 15 minutes later they lose sight of each other.

If the visibility at sea that day was 5 km, determine the constant speed of the boat travelling west.

Solution

We will call the boat travelling east Boat A, and the boat travelling west Boat B.

Boat A is travelling at a constant speed of 8 km/h.

Using the formula, distance = speed × time, in 15 minutes Boat A will travel
$8\,\dfrac{\text{km}}{\text{h}} \times \dfrac{15}{60}\text{ h} = 2\text{ km}$.

The visibility at sea is 5 km. Thus, Boat A and Boat B will be in sight of one another until they are 5 km apart. We are given that Boat A and Boat B are in sight of one another for 15 minutes. Thus, after 15 minutes Boat A and Boat B must be 5 km apart.

Since Boat A and Boat B start out 200 m = 0.2 km apart and Boat A travels 2 km in 15 minutes, Boat B must travel $5 - 0.2 - 2 = 2.8$ km in 15 minutes.

Since Boat B travelled 2.8 km in 15 minutes, using the formula speed = distance ÷ time, Boat B must have been travelling at a speed of
$2.8\text{ km} \div \dfrac{15}{60}\text{ h} = 2.8 \times \dfrac{60}{15} = 11.2\text{ km/h}$.

Therefore, Boat B was travelling at a speed of 11.2 km/h.

Problem of the Week
Problem C
Keep The Average

Jackie is making a sequence of six numbers using the following rules.

1. She choses any two numbers for the first two numbers.
2. The next four numbers are each the average of the previous two numbers.

After she creates the sequence, she tells her friend that the fourth number is 22 and the sixth number is 45. What numbers did Jackie choose for the first two numbers?

?	?	?	22	?	45

?	?	?	22	?	45

Problem of the Week
Problem C and Solution
Keep The Average

Problem

Jackie is making a sequence of six numbers using the following rules.

1. She choses any two numbers for the first two numbers.
2. The next four numbers are each the average of the previous two numbers.

After she creates the sequence, she tells her friend that the fourth number is 22 and the sixth number is 45. What numbers did Jackie choose for the first two numbers?

Solution

We give two solutions. Both will use the fact that if x is the average of two numbers y and z, then $\frac{y+z}{2} = x$, and it follows that $y + z = 2 \times x$.

Solution 1

In the first solution, we solve the problem by working backwards.

Since the sixth number in the sequence is equal to the average of the two previous numbers, the sixth number must be the average of the fourth and fifth numbers. So, the sum of the fourth and fifth numbers must be 2 times the sixth number, or $2 \times 45 = 90$. Therefore, the fifth number is $90 - 22 = 68$.

We now determine the third number. The fifth number in the sequence is the average of the third and fourth numbers. So, the sum of the third and fourth numbers is 2 times the fifth number, or $2 \times 68 = 136$. Therefore, the third number is $136 - 22 = 114$.

We now determine the second number. The fourth number in the sequence is the average of the second and third numbers. So, the sum of the second and third numbers is 2 times the fourth number, or $2 \times 22 = 44$. Therefore, the second number is $44 - 114 = -70$.

We now determine the first number. The third number in the sequence is the average of the first and second numbers. So, the sum of the first and second numbers is 2 times the third number, or $2 \times 114 = 228$. Therefore, the first number is $228 - (-70) = 228 + 70 = 298$.

Therefore, the first number is 298 and the second number is -70.

Solution 2

We will now present a similar, but more algebraic solution.

Let a represent the first number in the sequence, b represent the second number in the sequence, c represent the third number in the sequence, and d represent the fifth number in the sequence. We again solve this problem by working backwards.

Since the sixth number in the sequence is equal to the average of the fourth and fifth numbers, we have $45 = \frac{22+d}{2}$. Multiplying both sides by 2, we obtain $22 + d = 45 \times 2 = 90$. Rearranging, $d = 90 - 22 = 68$. Therefore, the fifth number in the sequence is 68.

We now determine the third number. Since the fifth number in the sequence is equal to the average of the third and fourth numbers, we have $68 = \frac{c+22}{2}$. Multiplying both sides by 2, we obtain $c + 22 = 68 \times 2 = 136$. Rearranging, $c = 136 - 22 = 114$. Therefore, the third number in the sequence is 114.

We now determine the second number. Since the fourth number in the sequence is equal to the average of the second and third numbers, we have $22 = \frac{b+114}{2}$. Multiplying both sides by 2, we obtain $b + 114 = 22 \times 2 = 44$. Rearranging, $b = 44 - 114 = -70$. Therefore, the second number in the sequence is -70.

We now determine the first number. Since the third number in the sequence is equal to the average of the first and second numbers, we have $114 = \frac{a+(-70)}{2}$. Multiplying both sides by 2, we obtain $a + (-70) = 114 \times 2 = 228$. Rearranging, $a = 228 + 70 = 298$. Therefore, the first number in the sequence is 298.

Therefore, the first number is 298 and the second number is -70.

298	-70	114	22	68	45

Problem of the Week

Problem C

Six Zeros

The product of the first seven positive integers is equal to

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Mathematicians will write this product as $7!$. This is read as "7 factorial". So, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

This factorial notation can be used with any positive integer. For example, $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$. The three dots "$\cdots$" represent the product of the integers between 9 and 3.

In general, for a positive integer n, $n!$ is equal to the product of the positive integers from 1 to n.

Find the smallest positive integer n such that $n!$ ends in exactly six zeros.

... 000 000

... 000 000

Problem of the Week
Problem C and Solution
Six Zeros

Problem

The product of the first seven positive integers is equal to

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In general, for a positive integer n, $n!$ is equal to the product of the positive integers from 1 to n.

Find the smallest positive integer n such that $n!$ ends in exactly six zeros.

Solution

We start by examining the first few factorials:

$$\begin{aligned}
1! &= 1 \\
2! &= 2 \times 1 = 2 \\
3! &= 3 \times 2 \times 1 = 6 \\
4! &= 4 \times 3 \times 2 \times 1 = 24 \\
5! &= 5 \times 4 \times 3 \times 2 \times 1 = \mathbf{120} \\
6! &= 6 \times (5 \times 4 \times 3 \times 2 \times 1) = 6 \times 5! = 6(120) = 720 \\
7! &= 7 \times (6 \times 5 \times 4 \times 3 \times 2 \times 1) = 7 \times 6! = 7(720) = 5040 \\
8! &= 8 \times (7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 8 \times 7! = 8(5040) = 40\,320 \\
9! &= 9 \times (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 9 \times 8! = 9(40\,320) = 362\,880 \\
10! &= 10 \times (9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 10 \times 9! = 10(362\,880) = \mathbf{3\,628\,800}
\end{aligned}$$

These numbers are getting very large and soon will not fit on the display of a standard calculator. So, let's look at what is going on.

We observe that 5! ends in 0 and 10! ends in 00. Notice that the number of zeros at the end of the number increased by one at each of 5! and at 10!. Why is this?

A zero is added to the end of a positive integer when we multiply by 10. Multiplying a number by 10 is the same as multiplying a number by 2 and then by 5, or by 5 and then by 2, since $2 \times 5 = 10$ and $5 \times 2 = 10$. We must determine the next time we multiply by 2 and 5 (in some order), to know the next time the number of zeros at the end of the number increases again. Every time we multiply by an even positive integer we are multiplying by at least one more 2. In the integers from 1 to n, there are less multiples of 5. So, each multiple of 5 will affect the number of zeros at the end of the product.

Multiplying by 11, 12, 13, and 14 increases the number of 2s we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 15 since $15 = 5 \times 3$. So 15! will end in exactly three zeros, 000.

Multiplying by 16, 17, 18, and 19 increases the number of 2s we multiply by but not the number of 5s. So the number of zeros at the end of the product does not change. The next time we multiply by a 5 is when we multiply by 20 since $20 = 4 \times 5$. So 20! will end in exactly four zeros, 0000.

Multiplying by 21, 22, 23, and 24 increases the number of 2s we multiply by but not the number of 5s. The next time we multiply by a 5 is when we multiply by 25. In fact, multiplying by 25 is the same as multiplying by 5 twice since $25 = 5 \times 5$. So when we multiply by 25, we will increase the number of zeros on the end of the product by two. So 25! will end in exactly six zeros, 000 000.

Therefore, the smallest positive integer n such that $n!$ ends in exactly six zeros is 25. (It could be noted that 26!, 27!, 28!, and 29! also end in six zeros.)

For the curious,

$$24! = 620\,448\,401\,733\,239\,439\,360\,000$$

and

$$25! = 15\,511\,210\,043\,330\,985\,984\,000\,000$$

Problem of the Week
Problem C
Thelma's Chips

Thelma has two piles of bingo chips. In each pile there are green and yellow chips. In one pile, the ratio of the number of green chips to the number of yellow chips is 1 : 2. In the second pile, the ratio of the number of green chips to the number of yellow chips is 3 : 5. If Thelma has a total of 20 green chips, then determine the possibilities for the total number of yellow chips.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Thelma's Chips

Problem

Thelma has two piles of bingo chips. In each pile there are green and yellow chips. In one pile, the ratio of the number of green chips to the number of yellow chips is $1 : 2$. In the second pile, the ratio of the number of green chips to the number of yellow chips is $3 : 5$. If Thelma has a total of 20 green chips, then determine the possibilities for the total number of yellow chips.

Solution

Solution 1

In this solution, we first look at all possible combinations of green and yellow chips in the second pile. Since the ratio of the number of green chips to the number of yellow chips in the second pile is $3 : 5$, we know that the number of green chips in this second pile must be a positive multiple of 3. We also know that there are at most 20 green chips in this pile. Thus, the only possible values for the number of green chips in the second pile are 3, 6, 9, 12, 15, and 18. Then, using the fact that the ratio of the number of green chips to the number of yellow chips is $3 : 5$, we can determine the number of yellow chips in the second pile for each case. We can also determine the number of green chips in the first pile by subtracting the number of green chips in the second pile from 20. Finally, we can determine the number of yellow chips in the first pile by multiplying the number of green chips in the first pile by 2. This information for each case is summarized in the table below.

Number of green chips in pile 2	Number of yellow chips in pile 2	Number of green chips in pile 1	Number of yellow chips in pile 1	Total number of yellow chips
3	5	$20 - 3 = 17$	34	$5 + 35 = 39$
6	10	$20 - 6 = 14$	28	$10 + 28 = 38$
9	15	$20 - 9 = 11$	22	$15 + 22 = 37$
12	20	$20 - 12 = 8$	16	$20 + 16 = 36$
15	25	$20 - 15 = 5$	10	$25 + 10 = 35$
18	30	$20 - 18 = 2$	4	$30 + 4 = 34$

Therefore, there are six possible values for the total number of yellow chips. There could be 34, 35, 36, 37, 38, or 39 yellow chips in total.

Solution 2

Let a represent the number of green chips in the first pile, where a is a positive integer. Since the ratio of green chips to yellow chips in this pile is $1 : 2$, then there are $2a$ yellow chips in this pile.

Let $3b$ represent the number of green chips in the second pile, where b is a positive integer. Since the ratio of green chips to yellow chips in this pile is $3 : 5$, then there are $5b$ yellow chips in this pile.

In total, there are 20 green chips, so $a + 3b = 20$. Also, the total number of yellow chips is equal to $2a + 5b$.

We consider all the possible values for positive integers a and b that satisfy the equation $a + 3b = 20$. Using these values of a and b, we can then find the possible values of $2a + 5b$, and hence the possible values for the total number of yellow chips.

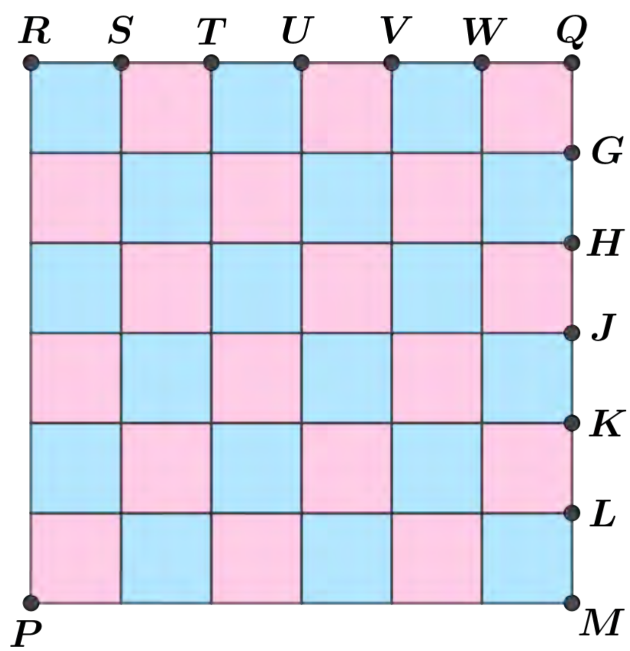
The results are summarized in the table below.

$a + 3b$	b	a	$2a + 5b$
20	1	17	39
20	2	14	38
20	3	11	37
20	4	8	36
20	5	5	35
20	6	2	34

Therefore, there are six possible values for the total number of yellow chips. There could be 34, 35, 36, 37, 38, or 39 yellow chips in total.

Problem of the Week
Problem C
All Equal

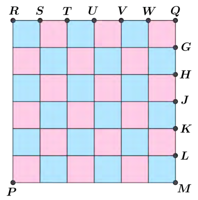
Using two cuts, we want to divide the 6 m by 6 m grid shown into three regions of equal area.



One way to do so is by making a horizontal cut through H and a second horizontal cut through K. This method of cutting the grid works, but is not very creative.

To make things a little more interesting, we must still make two straight cuts, but each cut must start at point P. Each of these two cuts will pass through a point on the outer perimeter of the grid.

Find the length of each cut. Round your answer to one decimal.



Problem of the Week

Problem C and Solution

All Equal

Problem

Using two cuts, we want to divide the 6 m by 6 m grid shown into three regions of equal area.

One way to do so is by making a horizontal cut through H and a second horizontal cut through K. This method of cutting the grid works, but is not very creative.

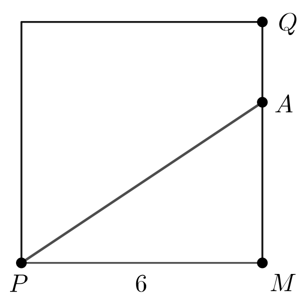
To make things a little more interesting, we must still make two straight cuts, but each cut must start at point P. Each of these two cuts will pass through a point on the outer perimeter of the grid.

Find the length of each cut. Round your answer to one decimal.

Solution

The area of the entire 6 m by 6 m square grid is $6 \times 6 = 36 \text{ m}^2$. Since the square is divided into three regions of equal area, the area of each region must be $\frac{36}{3} = 12 \text{ m}^2$.

Consider the line through P that passes through some point on side QM. Let A be the point where this line intersects QM.



Since $\angle PMQ = 90^\circ$, $\triangle PMA$ is a right-angled triangle with base $PM = 6$ m and height MA.

Using the formula area $= \frac{\text{base} \times \text{height}}{2}$, we have area of $\triangle PMA = \frac{6 \times MA}{2} = 3 \times MA$.

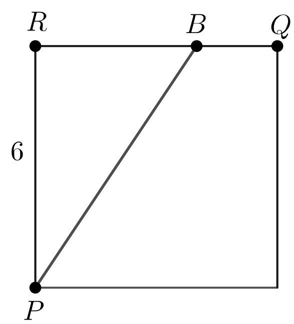
We need the area of $\triangle PMA$ to be 12 m^2. Therefore, $3 \times MA = 12$, and so $MA = 4$ m. Since H is the point on QM with $MH = 4$ m, we must have $A = H$. Therefore, one line passes through the point H.

Since $\triangle PMA$ is a right-angled triangle, using the Pythagorean Theorem we have

$$\begin{aligned} PA^2 &= PM^2 + MA^2 \\ &= 6^2 + 4^2 \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

Therefore, $PA = \sqrt{52} \approx 7.2$, since $PA > 0$.

Consider the line through P that passes through some point on side RQ. Let B be the point where this line intersects RQ.



Since $\angle PRQ = 90°$, $\triangle PRB$ is a right-angled triangle with height $PR = 6$ m and base RB.

Using the formula area $= \dfrac{\text{base} \times \text{height}}{2}$, we have area of $\triangle PRB = \dfrac{RB \times 6}{2} = 3 \times RB$.

We need the area of $\triangle PRB$ to be 12 m^2. Therefore, $3 \times RB = 12$, and so $RB = 4$ m. Since V is the point on RQ with $RV = 4$ m, we must have $B = V$. Therefore, the other line passes through the point V.

Therefore, one line passes through point H and the other passes through point V.

Since $\triangle PRB$ is a right-angled triangle, using the Pythagorean Theorem we have

$$\begin{aligned} PB^2 &= PR^2 + RB^2 \\ &= 6^2 + 4^2 \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

Therefore, $PB = \sqrt{52} \approx 7.2$, since $PB > 0$.

Therefore, the length of each cut is approximately 7.2 m.

Extension:

Try dividing the grid into three regions of equal area using three cuts. (Each cut does not necessarily need to be to the outer perimeter of the grid.)

Problem of the Week
Problem C
Two Numbers In

The POTW Input/Output Machine takes a number as input and adds 10 to the number. The machine then takes this sum and multiplies it by 2. Finally, the machine takes this product, subtracts 30 from the number, and outputs this new number.

Anala and Mei each input a positive integer into the machine. If the sum of their two outputs is 130, how many possibilities are there for the positive integer that Anala input into the machine?

Problem of the Week

Problem C and Solution

Two Numbers In

Problem

The POTW Input/Output Machine takes a number as input and adds 10 to the number. The machine then takes this sum and multiplies it by 2. Finally, the machine takes this product, subtracts 30 from the number, and outputs this new number.

Anala and Mei each input a positive integer into the machine. If the sum of their two outputs is 130, how many possibilities are there for the positive integer that Anala input into the machine?

Solution

We will work backward from the final sum, 130, by 'undoing' each of the three operations to determine the sum of their two numbers before any operations were performed.

The final operation performed by the machine on each number was to subtract 30. Subtracting 30 from each number decreases their sum by 60. Therefore, the sum of the two numbers immediately before the third operation was performed was $130 + 60 = 190$.

Multiplying each of their numbers by 2 increases the sum of the two numbers by a factor of 2. Since the second sum of their two numbers was 190, the sum of their two numbers immediately before the second operation was performed must have been $180 \div 2 = 95$.

Finally, the first operation performed by each of Anala and Mei was to add 10 to their number. Adding 10 to each of their numbers increases the sum by 20, and so the sum of their numbers before the first operation must have been $95 - 20 = 75$.

Each of their original integers are positive and the two integers have a sum of 75.

Therefore, Anala's original integer could be any integer from 1 to 74, inclusive. Thus, there are 74 possibilities for Anala's original integer.

Geometry & Measurement (G)

TAKE ME TO THE COVER

Problem of the Week

Problem C

Two Squares

Simone has a rope that is 60 cm long. They cut the rope into two pieces so that the ratio of the lengths of the two pieces is $7 : 3$. Each piece of the rope is then arranged, with its two ends touching, to form a square.

What is the total area of the two squares?

Themes Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Two Squares

Problem

Simone has a rope that is 60 cm long. They cut the rope into two pieces so that the ratio of the lengths of the two pieces is $7 : 3$. Each piece of the rope is then arranged, with its two ends touching, to form a square.

What is the total area of the two squares?

Solution

Since the rope is cut in the ratio of $7 : 3$, the ratio of the longer piece to the whole rope will be $7 : (7 + 3)$ or $7 : 10$. This means the length of the longer piece will be $\frac{7}{10}$ of the length of the whole rope. Similarly, the length of the shorter piece would be $\frac{3}{10}$ of the length of the whole rope. Therefore, the longer piece is $\frac{7}{10}$ of 60 or $\frac{7}{10} \times 60 = 42$ cm long. Also, the shorter piece is $\frac{3}{10}$ of 60 or $\frac{3}{10} \times 60 = 18$ cm long.

Each of the two pieces is then used to form a square. The perimeter of each square is the length of the rope used to form it. The side length of the longer square is $42 \div 4 = 10.5$ cm and the side length of the shorter square is $18 \div 4 = 4.5$ cm.

To find the area of each square, we multiply the length by the width. In effect, to find the area of the square, we square the side length. Thus, the area of the larger square is $10.5 \times 10.5 = 10.5^2 = 110.25\text{ cm}^2$ and the area of the smaller square is $4.5 \times 4.5 = 4.5^2 = 20.25\text{ cm}^2$.

Therefore, the total area of the two squares is $110.25 + 20.25 = 130.5\text{ cm}^2$.

For Further Thought:

The ratio of the area of the larger square to the area of the smaller square is

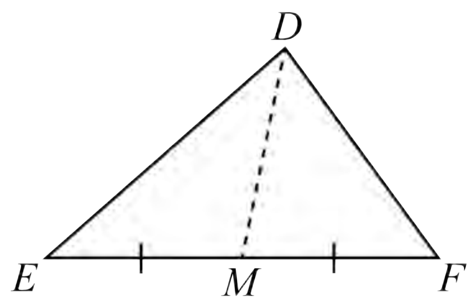
$$110.25 : 20.25 = 11\,025 : 2025 = 441 : 81 = 49 : 9 = 7^2 : 3^2$$

Notice that the ratio of the perimeter of the larger square to the perimeter of the smaller square is $7 : 3$ and the ratio of their areas is $7^2 : 3^2$. In general, if the ratio of the perimeters of two squares is $a : b$, is it true that the ratio of the areas of the two squares is $a^2 : b^2$?

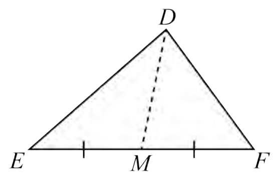
Problem of the Week
Problem C
Three Perimeters

A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle DEF$, a median is drawn from vertex D, meeting side EF at point M.



The perimeter of $\triangle DEF$ is 24. The perimeter of $\triangle DEM$ is 18. The perimeter $\triangle DFM$ is 16. Determine the length of the median DM.



Problem of the Week

Problem C and Solution

Three Perimeters

Problem

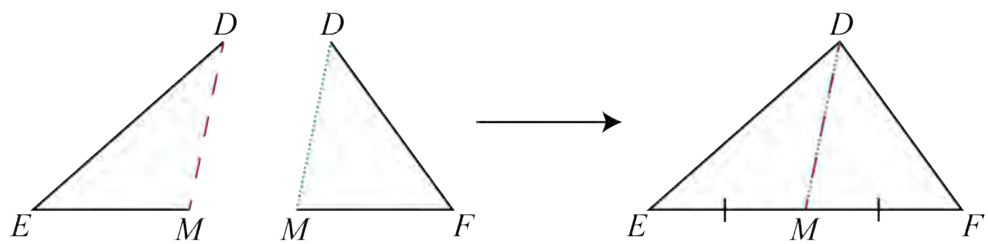
A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle DEF$, a median is drawn from vertex D, meeting side EF at point M. The perimeter of $\triangle DEF$ is 24. The perimeter of $\triangle DEM$ is 18. The perimeter $\triangle DFM$ is 16. Determine the length of the median DM.

Solution

Solution 1

The perimeter of a triangle is equal to the sum of its three side lengths. Notice that the length of side EF is equal to the sum of the lengths of sides EM and MF. It follows that when we combine the perimeters of $\triangle DEM$ and $\triangle DFM$, we obtain the perimeter of $\triangle DEF$ plus two lengths of the median DM.



In other words, since the perimeter of $\triangle DEM$ is 18, the perimeter of $\triangle DFM$ is 16, and the perimeter of $\triangle DEF$ is 24, it follows that $18 + 16 = 24 + 2 \times DM$. Then $34 = 24 + 2 \times DM$, and so $2 \times DM = 10$. Therefore, the length of the median DM is 5.

Solution 2

In this solution, we take a more algebraic approach to solving the problem, using more formal equation solving.

Let $DE = t$, $EM = p$, $MF = q$, $DF = r$, and $DM = m$.

Since the perimeter of $\triangle DEM$ is 18, we can write the following equation.

$$t + p + m = 18$$

$$t + p = 18 - m \tag{1}$$

Since the perimeter of $\triangle DFM$ is 16, we can write the following equation.

$$\begin{aligned} q + r + m &= 16 \\ q + r &= 16 - m \qquad (2) \end{aligned}$$

Since the perimeter of $\triangle DEF$ is 24, we can write the following equation.

$$t + p + q + r = 24 \qquad (3)$$

Adding equations (1) and (2) gives the following.

$$\begin{aligned} t + p &= 18 - m \qquad (1) \\ q + r &= 16 - m \qquad (2) \\ t + p + q + r &= 18 - m + 16 - m \end{aligned}$$

However from equation (3), we know that $t + p + q + r = 24$. So we can write and solve the following equation.

$$\begin{aligned} 18 - m + 16 - m &= 24 \\ 34 - 2m &= 24 \\ -2m &= 24 - 34 \\ -2m &= -10 \\ \frac{-2m}{-2} &= \frac{-10}{-2} \\ m &= 5 \end{aligned}$$

Therefore, the length of the median DM is 5.

Extension:

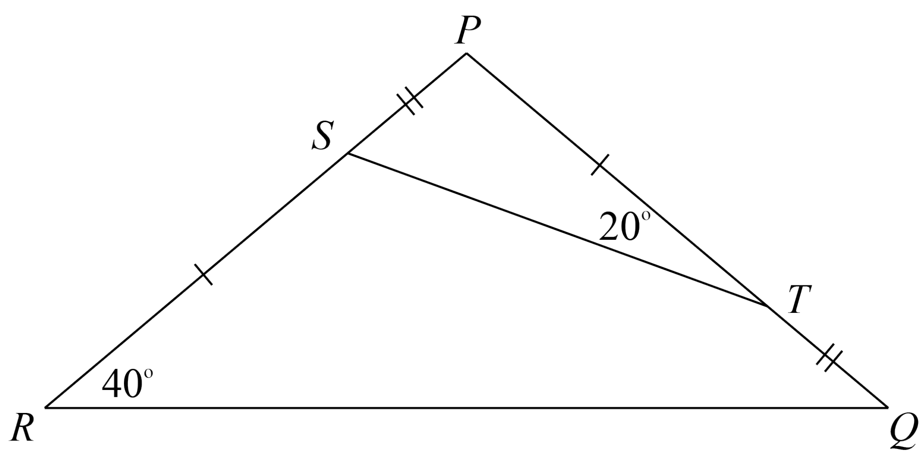
In the solution we never used the fact that DM is a median and that $EM = MF$. This means that there could be other triangles that satisfy the conditions of the problem without DM being the median. Indeed there are! Try creating a few different triangles with $DM = 5$ that satisfy all the conditions of the problem except the condition that DM is a median. You can do this using manipulatives, geometry software, or by hand. However, you may need some high school mathematics to calculate the precise dimensions.

It turns out that there is only one triangle that satisfies all the conditions of the problem including the fact that DM is a median.

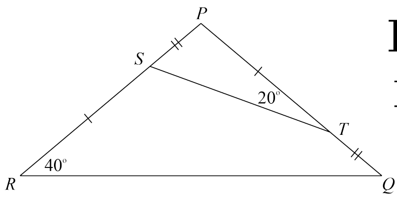
Problem of the Week
Problem C
The Missing Pieces

The following information is known about $\triangle PQR$.

- The point S is on side PR and the point T is on side PQ.
- The distance from P to S is equal to the distance from T to Q.
- The distance from S to R is equal to the distance from P to T.
- $\angle PRQ = 40^\circ$ and $\angle PTS = 20^\circ$.



Determine the value of each of the five other interior angles. That is, determine the values of $\angle RPQ$, $\angle STQ$, $\angle TQR$, $\angle RST$, and $\angle PST$.



Problem of the Week
Problem C and Solution
The Missing Pieces

Problem

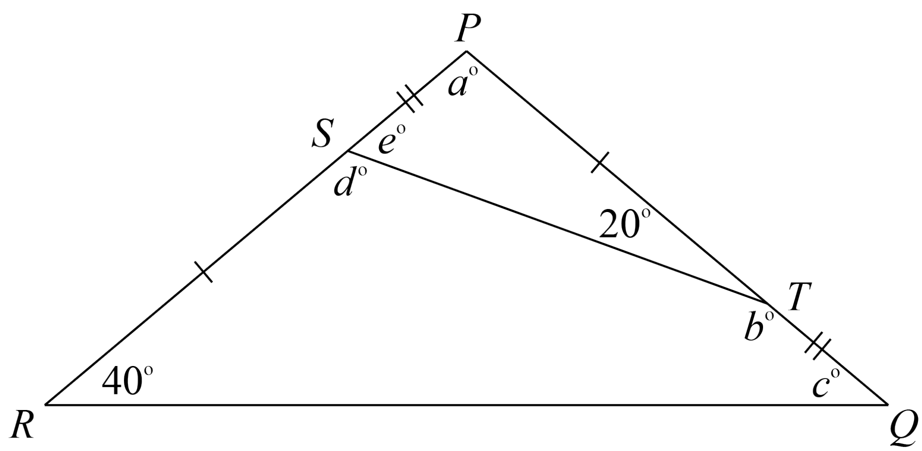
The following information is known about $\triangle PQR$.

- The point S is on side PR and the point T is on side PQ.
- The distance from P to S is equal to the distance from T to Q.
- The distance from S to R is equal to the distance from P to T.
- $\angle PRQ = 40^\circ$ and $\angle PTS = 20^\circ$.

Determine the value of each of the five other interior angles. That is, determine the values of $\angle RPQ$, $\angle STQ$, $\angle TQR$, $\angle RST$, and $\angle PST$.

Solution

First, we let $\angle RPQ$ measure a°, $\angle STQ$ measure b°, $\angle TQR$ measure c°, $\angle RST$ measure d°, and $\angle PST$ measure e°.



Since $\angle PTQ$ is a straight angle, $20 + b = 180$, and so $b = 160$.

Since $PS = TQ$ and $SR = PT$, it follows that $PS + PR = PT + TQ$, and so $PR = PQ$ and $\triangle PQR$ is isosceles. Therefore $\angle PRQ = \angle PQR$, and so $c = 40$.

Since the angles in a triangle sum to 180°, in $\triangle PQR$,

$$\begin{aligned} a + 40 + c &= 180 \\ a + 40 + 40 &= 180 \\ a + 80 &= 180 \\ a &= 100 \end{aligned}$$

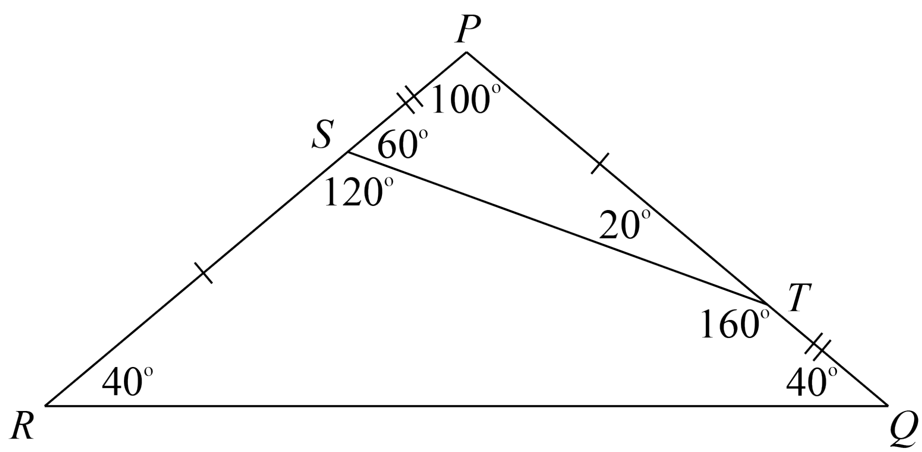
Similarly, in $\triangle PST$,

$$\begin{aligned} a + e + 20 &= 180 \\ 100 + e + 20 &= 180 \\ 120 + e &= 180 \\ e &= 60 \end{aligned}$$

Since $\angle PSR$ is a straight angle,

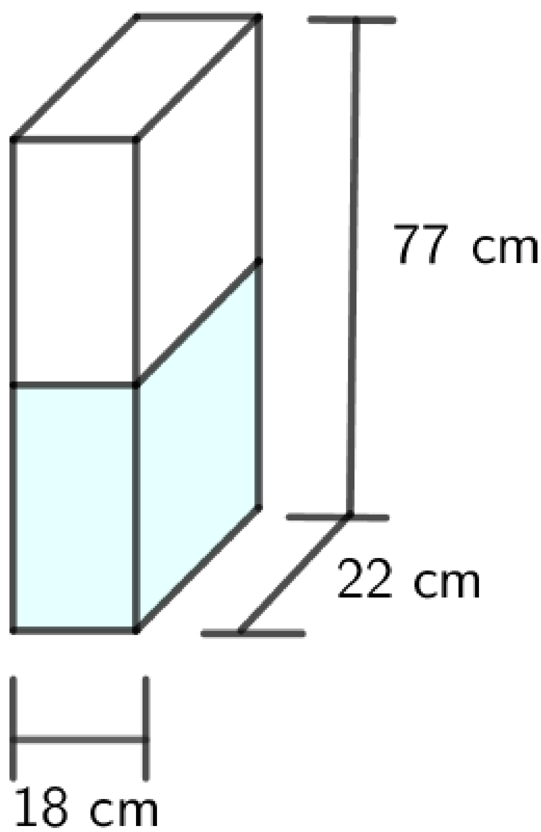
$$\begin{aligned} e + d &= 180 \\ 60 + d &= 180 \\ d &= 120 \end{aligned}$$

We have determined the value of all the other five interior angles.
$\angle RPQ = a^\circ = 100^\circ$, $\angle STQ = b^\circ = 160^\circ$, $\angle TQR = c^\circ = 40^\circ$,
$\angle RST = d^\circ = 120^\circ$, and $\angle PST = e^\circ = 60^\circ$.

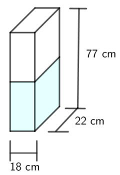


Problem of the Week
Problem C
Ice Box

A metal box in the form of a rectangular prism has an 18 cm by 22 cm base and a height of 77 cm. The box is to be filled with water, which will then be frozen. When water freezes it expands by approximately 10%. Determine the maximum depth to which the box can be filled with water so that when the water freezes, the ice does not go above the top of the container.



Themes Geometry & Measurement, Number Sense



Problem of the Week
Problem C and Solution
Ice Box

Problem

A metal box in the form of a rectangular prism has an 18 cm by 22 cm base and a height of 77 cm. The box is to be filled with water, which will then be frozen. When water freezes it expands by approximately 10%. Determine the maximum depth to which the box can be filled with water so that when the water freezes, the ice does not go above the top of the container.

Solution

Solution 1

To determine the volume of a rectangular prism, we multiply its length, width, and height together. So, the maximum volume of the metal box is

$$18 \times 22 \times 77 = 30\,492 \text{ cm}^3$$

Let the original depth of water in the metal box be h cm.

The water volume before freezing is $18 \times 22 \times h = (396 \times h)$ cm^3. After the water freezes, the volume increases by 10% to 110% of its current volume. That is, after freezing the volume is

$$110\% \text{ of } 396 \times h = 1.1 \times 396 \times h = (435.6 \times h) \text{ cm}^3$$

But the volume after freezing is the maximum volume, $30\,492$ cm^3. Therefore, $435.6 \times h = 30\,492$ and it follows that $h = 30\,492 \div 435.6 = 70$ cm.

Therefore, the maximum depth to which the box can be filled is 70 cm.

Solution 2

In this solution we note that the length and width remain the same in the volume calculations before and after the water freezes. We need only concern ourselves with the change in the depth of the water.

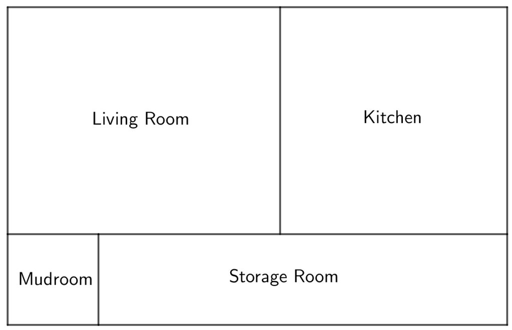
Let the original depth of water in the container be h cm.

After freezing, the depth increases by 10% to 110% of its depth before freezing. So, after freezing the depth will be 110% of $h = 1.1 \times h = 77$ cm, the maximum height of the container. Then $h = 77 \div 1.1 = 70$ cm.

Therefore, the maximum depth to which the box can be filled is 70 cm.

Problem of the Week
Problem C
Dollhouse

The first level of a dollhouse is in the shape of a rectangle. Its floor plan is shown in the following diagram.



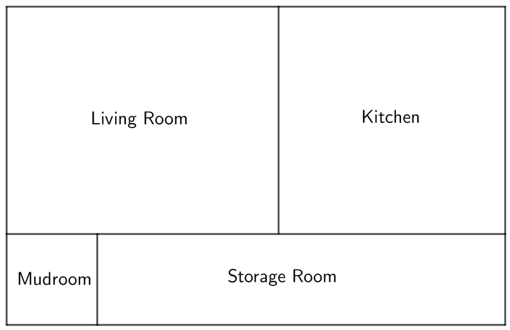
Both the mudroom and the kitchen are square with areas of 400 cm^2 and 2500 cm^2, respectively. The living room is rectangular with an area of 3000 cm^2.

Determine the area of the rectangular storage room.

Themes Geometry & Measurement, Number Sense

Problem of the Week
Problem C and Solution
Dollhouse

Problem
The first level of a dollhouse is in the shape of a rectangle. Its floor plan is shown in the following diagram.



Both the mudroom and the kitchen are square with areas of 400 cm^2 and 2500 cm^2, respectively. The living room is rectangular with an area of 3000 cm^2.

Determine the area of the rectangular storage room.

Solution
Let the width of a room be the vertical length of the room on the diagram. Let the length of a room be the horizontal length of the room on the diagram.

The kitchen is a square and has an area of 2500 cm^2. Its length and width must both be 50 cm since $50 \times 50 = 2500$ cm^2. The living room and kitchen have the same width. So the width of the living room must also be 50 cm. But the area of the living room is 3000 cm^2, so the length of the living room is 60 cm since $50 \times 60 = 3000$ cm^2.

The mudroom is a square and has an area of 400 cm^2. Its length and width must both be 20 cm since $20 \times 20 = 400$ cm^2. The mudroom and storage room have the same width. So the width of the storage room must also be 20 cm.

Now the length of the whole house can be calculated in two ways. We will equate these two expressions to find the length of the storage room.

$$\begin{aligned}
\text{mudroom length} + \text{storage room length} &= \text{living room length} + \text{kitchen length} \\
20 + \text{storage room length} &= 60 + 50 \\
20 + \text{storage room length} &= 110 \\
\text{storage room length} &= 90 \text{ cm}
\end{aligned}$$

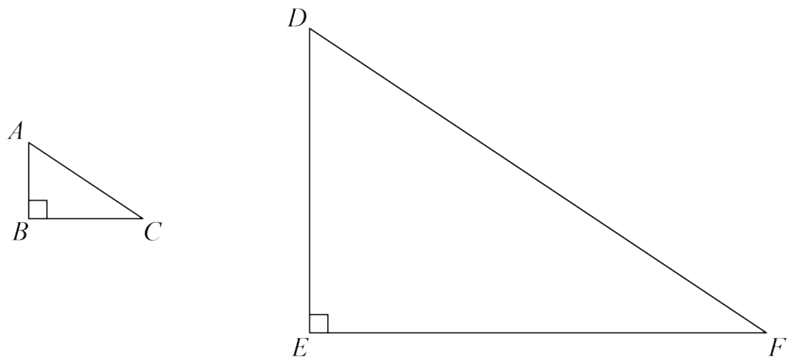
Since the width of the storage room is 20 cm and the length of the storage room is 90 cm, the area of the storage room is $20 \times 90 = 1800$ cm^2.

Problem of the Week

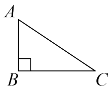
Problem C

A Bigger Triangle

Naveen drew a right-angled triangle, $\triangle ABC$, with an area of 14 cm^2. His brother Anand drew a bigger right-angled triangle, $\triangle DEF$, with side lengths four times the lengths of the sides in $\triangle ABC$. In particular, $DE = 4 \times AB$, $EF = 4 \times BC$, and $DF = 4 \times AC$.



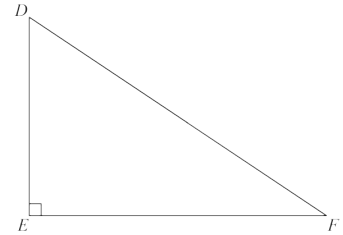
Calculate the area of $\triangle DEF$.



Problem of the Week

Problem C and Solution

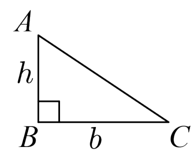
A Bigger Triangle



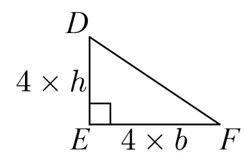
Problem

Naveen drew a right-angled triangle, $\triangle ABC$, with an area of 14 cm^2. His brother Anand drew a bigger right-angled triangle, $\triangle DEF$, with side lengths four times the lengths of the sides in $\triangle ABC$. In particular, $DE = 4 \times AB$, $EF = 4 \times BC$, and $DF = 4 \times AC$. Calculate the area of $\triangle DEF$.

Solution



In $\triangle ABC$, let b represent the length of the base, BC, and h represent the length of the height, AB. Then the area of $\triangle ABC$ is equal to $\frac{b\times h}{2}$. We know this area is equal to 14 cm^2, so it follows that $14 = \frac{b\times h}{2}$, or $28 = b \times h$.



$\triangle DEF$ is formed by multiplying each of the side lengths of $\triangle ABC$ by 4. So the length of the base of $\triangle DEF$ is equal to $4 \times b$ and the length of the height is equal to $4 \times h$. We can calculate the area of $\triangle DEF$ as follows.

$$\begin{aligned}
\text{area of } \triangle DEF &= \frac{(4 \times b) \times (4 \times h)}{2} \\
&= \frac{16 \times b \times h}{2} \\
&= \frac{16 \times 28}{2}, \text{ since } b \times h = 28 \\
&= 224
\end{aligned}$$

Therefore, the area of $\triangle DEF$ is 224 cm^2.

Extension:

Notice that $\triangle DEF$ has side lengths that are each 4 times the corresponding side lengths of $\triangle ABC$ and that the area of $\triangle DEF$ ended up being $224 = 16 \times 14 = 4^2 \times$ area of $\triangle ABC$.

Show that if $\triangle DEF$ has side lengths that are each k times the corresponding side lengths of $\triangle ABC$, then the area of $\triangle DEF$ will be equal to k^2 times the area of $\triangle ABC$.

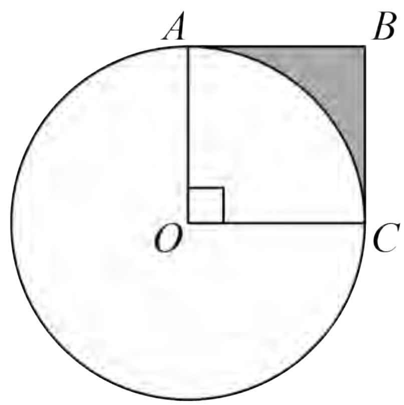
Problem of the Week

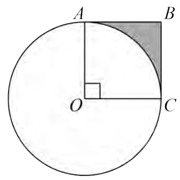
Problem C

Just Outside

In square $OABC$, points A and C lie on the circumference of a circle with centre O, and B lies outside of the circle. Square $OABC$ has an area of $36\,\text{m}^2$.

Determine the area of the shaded region inside square $OABC$ and outside the circle with centre O, rounded to two decimal places.





Problem of the Week

Problem C and Solution

Just Outside

Problem

In square $OABC$, points A and C lie on the circumference of a circle with centre O, and B lies outside of the circle. Square $OABC$ has an area of $36\,\text{m}^2$.

Determine the area of the shaded region inside square $OABC$ and outside the circle with centre O, rounded to two decimal places.

Solution

Since $OABC$ is a square with an area of $36\,\text{m}^2$, its side length must be $6\,\text{m}$. That is, $OA = OC = 6\,\text{m}$.

Since A lies on the circumference of the circle with centre O, the radius of the circle is $r = OA = 6\,\text{m}$.

Therefore, the area of the circle is $\pi \times r^2 = \pi \times 6^2 = 36\pi\,\text{m}^2$.

Since $OABC$ is a square, $\angle AOC = 90°$.

Therefore, the area of sector OAC is $\frac{90°}{360°} = \frac{1}{4}$ of the area of the circle.

In other words, the area of the sector OAC is $\frac{1}{4} \times 36\pi = 9\pi\,\text{m}^2$.

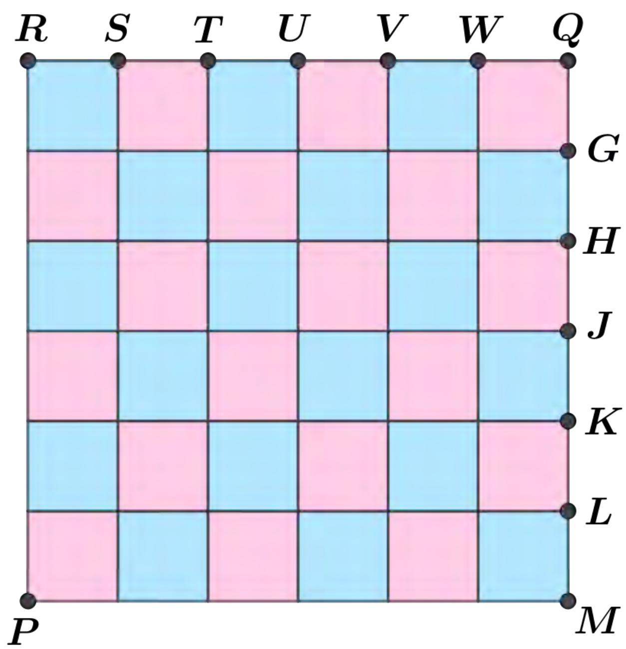
Therefore,

$$\begin{aligned}\text{Area of shaded region} &= \text{Area of square } OABC - \text{Area of sector } OAC \\ &= 36 - 9\pi \\ &\approx 7.73\,\text{m}^2\end{aligned}$$

Note: In the problem you were asked to give your answer rounded to two decimal places. However, many times in mathematics we are actually interested in the *exact* answer. In this case, the exact answer would be $(36 - 9\pi)\,\text{m}^2$.

Problem of the Week
Problem C
All Equal

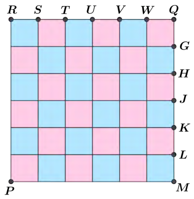
Using two cuts, we want to divide the 6 m by 6 m grid shown into three regions of equal area.



One way to do so is by making a horizontal cut through H and a second horizontal cut through K. This method of cutting the grid works, but is not very creative.

To make things a little more interesting, we must still make two straight cuts, but each cut must start at point P. Each of these two cuts will pass through a point on the outer perimeter of the grid.

Find the length of each cut. Round your answer to one decimal.



Problem of the Week

Problem C and Solution

All Equal

Problem

Using two cuts, we want to divide the 6 m by 6 m grid shown into three regions of equal area.

One way to do so is by making a horizontal cut through H and a second horizontal cut through K. This method of cutting the grid works, but is not very creative.

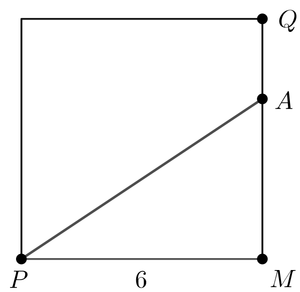
To make things a little more interesting, we must still make two straight cuts, but each cut must start at point P. Each of these two cuts will pass through a point on the outer perimeter of the grid.

Find the length of each cut. Round your answer to one decimal.

Solution

The area of the entire 6 m by 6 m square grid is $6 \times 6 = 36$ m^2. Since the square is divided into three regions of equal area, the area of each region must be $\dfrac{36}{3} = 12$ m^2.

Consider the line through P that passes through some point on side QM. Let A be the point where this line intersects QM.



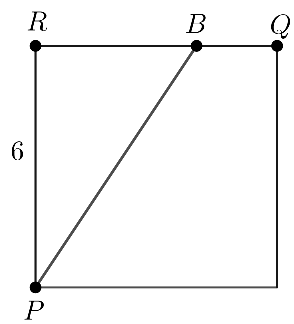
Since $\angle PMQ = 90^\circ$, $\triangle PMA$ is a right-angled triangle with base $PM = 6$ m and height MA.
Using the formula area $= \dfrac{\text{base} \times \text{height}}{2}$, we have area of $\triangle PMA = \dfrac{6 \times MA}{2} = 3 \times MA$.
We need the area of $\triangle PMA$ to be 12 m^2. Therefore, $3 \times MA = 12$, and so $MA = 4$ m. Since H is the point on QM with $MH = 4$ m, we must have $A = H$. Therefore, one line passes through the point H.

Since $\triangle PMA$ is a right-angled triangle, using the Pythagorean Theorem we have

$$\begin{aligned} PA^2 &= PM^2 + MA^2 \\ &= 6^2 + 4^2 \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

Therefore, $PA = \sqrt{52} \approx 7.2$, since $PA > 0$.

Consider the line through P that passes through some point on side RQ. Let B be the point where this line intersects RQ.



Since $\angle PRQ = 90°$, $\triangle PRB$ is a right-angled triangle with height $PR = 6$ m and base RB.

Using the formula area $= \dfrac{\text{base} \times \text{height}}{2}$, we have area of $\triangle PRB = \dfrac{RB \times 6}{2} = 3 \times RB$.

We need the area of $\triangle PRB$ to be 12 m^2. Therefore, $3 \times RB = 12$, and so $RB = 4$ m. Since V is the point on RQ with $RV = 4$ m, we must have $B = V$. Therefore, the other line passes through the point V.

Therefore, one line passes through point H and the other passes through point V.

Since $\triangle PRB$ is a right-angled triangle, using the Pythagorean Theorem we have

$$\begin{aligned} PB^2 &= PR^2 + RB^2 \\ &= 6^2 + 4^2 \\ &= 36 + 16 \\ &= 52 \end{aligned}$$

Therefore, $PB = \sqrt{52} \approx 7.2$, since $PB > 0$.

Therefore, the length of each cut is approximately 7.2 m.

EXTENSION:

Try dividing the grid into three regions of equal area using three cuts. (Each cut does not necessarily need to be to the outer perimeter of the grid.)

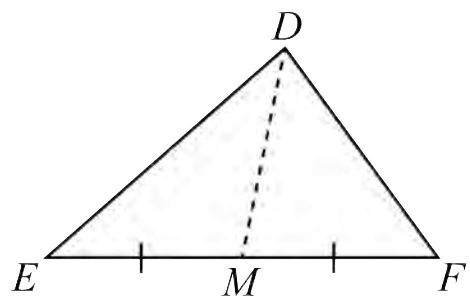
Algebra (A)

TAKE ME TO THE COVER

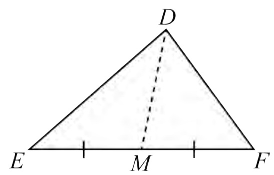
Problem of the Week
Problem C
Three Perimeters

A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle DEF$, a median is drawn from vertex D, meeting side EF at point M.



The perimeter of $\triangle DEF$ is 24. The perimeter of $\triangle DEM$ is 18. The perimeter $\triangle DFM$ is 16. Determine the length of the median DM.



Problem of the Week

Problem C and Solution

Three Perimeters

Problem

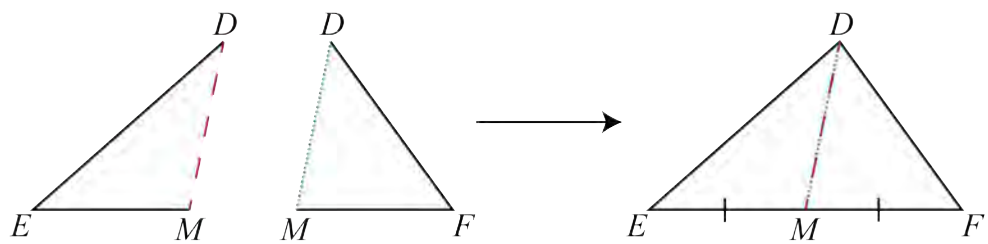
A *median* is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In $\triangle DEF$, a median is drawn from vertex D, meeting side EF at point M. The perimeter of $\triangle DEF$ is 24. The perimeter of $\triangle DEM$ is 18. The perimeter $\triangle DFM$ is 16. Determine the length of the median DM.

Solution

Solution 1

The perimeter of a triangle is equal to the sum of its three side lengths. Notice that the length of side EF is equal to the sum of the lengths of sides EM and MF. It follows that when we combine the perimeters of $\triangle DEM$ and $\triangle DFM$, we obtain the perimeter of $\triangle DEF$ plus two lengths of the median DM.



In other words, since the perimeter of $\triangle DEM$ is 18, the perimeter of $\triangle DFM$ is 16, and the perimeter of $\triangle DEF$ is 24, it follows that $18 + 16 = 24 + 2 \times DM$. Then $34 = 24 + 2 \times DM$, and so $2 \times DM = 10$. Therefore, the length of the median DM is 5.

Solution 2

In this solution, we take a more algebraic approach to solving the problem, using more formal equation solving.

Let $DE = t$, $EM = p$, $MF = q$, $DF = r$, and $DM = m$.

Since the perimeter of $\triangle DEM$ is 18, we can write the following equation.

$$t + p + m = 18$$

$$t + p = 18 - m \qquad (1)$$

Since the perimeter of $\triangle DFM$ is 16, we can write the following equation.

$$\begin{aligned} q + r + m &= 16 \\ q + r &= 16 - m \qquad (2) \end{aligned}$$

Since the perimeter of $\triangle DEF$ is 24, we can write the following equation.

$$t + p + q + r = 24 \qquad (3)$$

Adding equations (1) and (2) gives the following.

$$\begin{aligned} t + p &= 18 - m \qquad (1) \\ q + r &= 16 - m \qquad (2) \\ t + p + q + r &= 18 - m + 16 - m \end{aligned}$$

However from equation (3), we know that $t + p + q + r = 24$. So we can write and solve the following equation.

$$\begin{aligned} 18 - m + 16 - m &= 24 \\ 34 - 2m &= 24 \\ -2m &= 24 - 34 \\ -2m &= -10 \\ \frac{-2m}{-2} &= \frac{-10}{-2} \\ m &= 5 \end{aligned}$$

Therefore, the length of the median DM is 5.

Extension:

In the solution we never used the fact that DM is a median and that $EM = MF$. This means that there could be other triangles that satisfy the conditions of the problem without DM being the median. Indeed there are! Try creating a few different triangles with $DM = 5$ that satisfy all the conditions of the problem except the condition that DM is a median. You can do this using manipulatives, geometry software, or by hand. However, you may need some high school mathematics to calculate the precise dimensions.

It turns out that there is only one triangle that satisfies all the conditions of the problem including the fact that DM is a median.

Problem of the Week
Problem C
Gone Shopping

While grocery shopping, Terry has a way to approximate the total cost of his purchases. He simply approximates that each item will cost \$3.00.

One day, Terry purchased 20 items. He purchased items that each had an actual cost of either \$1.00, \$3.00, or \$7.50. Exactly seven of the purchased items had an actual cost of \$3.00. If the total actual cost of the 20 items was the same as the total approximated cost, how many items had an actual cost of \$7.50?

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Gone Shopping

Problem

While grocery shopping, Terry has a way to approximate the total cost of his purchases. He simply approximates that each item will cost \$3.00.

One day, Terry purchased 20 items. He purchased items that each had an actual cost of either \$1.00, \$3.00, or \$7.50. Exactly seven of the purchased items had an actual cost of \$3.00. If the total actual cost of the 20 items was the same as the total approximated cost, how many items had an actual cost of \$7.50?

Solution

The total approximated cost for the 20 items is $20 \times \$3 = \$60$. Since the total actual cost is the same as the total approximated cost, the total actual cost for the 20 items is \$60. Since 7 of the items cost \$3.00, it cost Terry $7 \times \$3 = \$21$ to buy these items. Therefore, the remaining $20 - 7 = 13$ items cost $\$60 - \$21 = \$39$.

From this point, we will continue with two different solutions.

Solution 1

In this solution, we will use systematic trial-and-error to solve the problem.

Let s represent the number of items Terry bought with an actual cost of \$7.50 and d represent the number of items that Terry bought with an actual cost \$1.00. Then the total cost of the \$7.50 items would be $7.5s$. Also, the total cost of the \$1.00 items would be $1d = d$. Since Terry's total remaining cost was \$39, then $7.5s + d = 39$. We also know that $s + d = 13$.

At this point we can systematically pick values for s and d that add to 13 and substitute into the equation $7.5s + d = 39$ to find the combination that works. (We can observe that $s < 6$ since $7.5 \times 6 = 45 > 39$. If this were the case, then d would have to be a negative number.)

Let's start with $s = 3$. Then $d = 13 - 3 = 10$. The cost of these items would be $7.5 \times 3 + 10 = 22.50 + 10 = \$32.50$, which is less than \$39.

So let's try $s = 4$. Then $d = 13 - 4 = 9$. The cost of these items would be $7.5 \times 4 + 9 = 30 + 9 = \$39$, which is the amount we want.

Therefore, Terry purchased 4 items that cost \$7.50.

Solution 2

In this solution, we will use algebra to solve the problem.

Let s represent the number of items that cost \$7.50. Therefore, $(13 - s)$ represents the number of items that cost \$1.00. Also, the total cost of the \$7.50 items would be $7.5s$, the total cost of the \$1.00 items would be $1 \times (13 - s) = 13 - s$, and the total of these two is $7.5s + 13 - s = 6.5s + 13$.

Since Terry's total remaining cost was \$39.00, we must have

$$\begin{aligned} 6.5s + 13 &= 39 \\ 6.5s + 13 - 13 &= 39 - 13 \\ 6.5s &= 26 \\ \frac{6.5s}{6.5} &= \frac{26}{6.5} \\ s &= 4 \end{aligned}$$

Therefore, Terry purchased 4 items that cost \$7.50.

Problem of the Week

Problem C

Teacher Road Trip 1

To help pass time on a long bus ride, 35 math teachers created a sequence of numbers, with each teacher saying one term in the sequence. The first teacher said the number 2, the second teacher said the number 8, and every teacher after that said the sum of the two previous terms. Thus,

- the third teacher said the sum of the first and second terms, which is $2 + 8 = 10$, and
- the fourth teacher said the sum of the second and third terms, which is $8 + 10 = 18$.

Once the final teacher said their number, the 25^{th} teacher announced they had made a mistake and their number should have been one more than what they had said. How much larger should the final teacher's number have been?

Problem of the Week
Problem C and Solution
Teacher Road Trip 1

Problem

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Once the final teacher said their number, the 25^{th} teacher announced they had made a mistake and their number should have been one more than what they had said. How much larger should the final teacher's number have been?

Solution

Solution 1

We will write out the sequence of numbers the teachers actually said, and then the sequence of numbers they should have said, and then find the difference between the last term in each sequence.

Here are the first 24 numbers that the teachers said:

2, 8, 10, 18, 28, 46, 74, 120, 194, 314, 508, 822, 1330, 2152, 3482, 5634, 9116, 14 750, 23 866, 38 616, 62 482, 101 098, 163 580, 264 678

Here are the correct 25^{th} to 35^{th} numbers that the teachers should have said:

428 258, 692 936, 1 121 194, 1 814 130, 2 935 324, 4 749 454, 7 684 778, 12 434 232, 20 119 010, 32 553 242, 52 672 252

Here are the 25^{th} to 35^{th} numbers that the teachers actually said:

428 257, 692 935, 1 121 192, 1 814 127, 2 935 319, 4 749 446, 7 684 765, 12 434 211, 20 118 976, 32 553 187, 52 672 163

The difference between the correct and incorrect 35^{th} number is $52\,672\,252 - 52\,672\,163 = 89$. Therefore, the 35^{th} number was off by 89, and so the final teacher's number should have been 89 larger than the number they had said.

Solution 2

In this solution we will solve the problem without actually calculating all the terms in the sequence.

We know the 25^{th} term is off by 1. Therefore, the next terms will be as follows.

- The 26^{th} term will also be off by 1 since it equals the sum of the 24^{th} term (which is unchanged) and the 25^{th} term (which is off by 1).
- The 27^{th} term will be off by 2 since it is the sum of the 25^{th} term (which is off by 1) and the 26^{th} term (which is off by 1).
- The 28^{th} term will be off by 3 since it is the sum of the 26^{th} term (which is off by 1) and the 27^{th} term (which is off by 2).

This pattern will continue on, so we can summarize it in a table.

Term Number	Amount Below the Correct Value
24	0
25	1
26	1
27	2
28	3
29	5
30	8
31	13
32	21
33	34
34	55
35	89

Therefore, the 35^{th} term was off by 89, and so the final teacher's number should have been 89 larger than the number they had said.

Notice that the terms in the right column of the table follow the same rule as the original question. That is, each term is the sum of the previous two terms.

For Further Thought: The last 11 numbers in the right column of the table are the first 11 numbers of a famous sequence known as the Fibonacci Sequence. You may wish to investigate the Fibonacci Sequence further.

Problem of the Week
Problem C
What's the Score?

In gym class, the yellow team and the blue team played soccer. Ali doesn't remember the final score of the game, but she does remember the following.

- There were six goals scored in total.
- Neither team scored more than two goals in a row at any point in the game.
- The blue team won the game.

Determine all the possible final scores and the different ways each score could have been obtained.

Problem of the Week
Problem C and Solution
What's the Score?

Problem

In gym class, the yellow team and the blue team played soccer. Ali doesn't remember the final score of the game, but she does remember the following.

- There were six goals scored in total.
- Neither team scored more than two goals in a row at any point in the game.
- The blue team won the game.

Determine all the possible final scores and the different ways each score could have been obtained.

Solution

In order to win, the blue team must have scored more goals than the yellow team. Since there were six goals scored in total, the only possibilities for the final scores are $4-2$, $5-1$, or $6-0$ for the blue team.

Next we need to check which of these scores are possible, given that neither team scored more than two goals in a row at any point in the game.

- Is a final score of $6-0$ possible?

  We can easily eliminate $6-0$, since the blue team would have had to score more than two goals in a row.

- Is a final score of $5-1$ possible?

  This would mean that the blue team scored 5 goals and the yellow team scored 1 goal. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team. These are all shown below.

  $YBBBBB$, $BYBBBB$, $BBYBBB$, $BBBYBB$, $BBBBYB$, $BBBBBY$

  As we can see, in all of these arrangements, the blue team scored more than two goals in a row. Thus, a final score of $5-1$ is not possible.

- Is a final score of $4-2$ possible?

  This would mean that the blue team scored 4 goals and the yellow team scored 2 goals. Is there a way to arrange these goals so that the blue team didn't score two goals in a row? Let's look at all the possible arrangements, where B represents a goal for the blue team, and Y represents a goal for the yellow team.

  - Case 1: The yellow team scored their 2 goals in a row. The possible arrangements are shown below.

    $$YYBBBB,\ BYYBBB,\ BBYYBB,\ BBBYYB,\ BBBBYY$$

    In this case, there is only 1 arrangement where neither team scored more than two goals in a row, namely $BBYYBB$.
  - Case 2: The yellow team did not score their 2 goals in a row. The possible arrangements are shown below.

    $$YBYBBB,\ YBBYBB,\ YBBBYB,\ YBBBBY,\ BYBYBB,$$
    $$BYBBYB,\ BYBBBY,\ BBYBYB,\ BBYBBY,\ BBBYBY$$

    In this case, there are 5 arrangements where neither team scored more than two goals in a row, namely
    $YBBYBB$, $BYBYBB$, $BYBBYB$, $BBYBYB$, and $BBYBBY$.

  Therefore, the only possible final score is $4-2$ for the blue team, and it could be obtained in the following six ways.

  $BBYYBB$, $YBBYBB$, $BYBYBB$, $BYBBYB$, $BBYBYB$, $BBYBBY$

Problem of the Week
Problem C
Thelma's Chips

Thelma has two piles of bingo chips. In each pile there are green and yellow chips. In one pile, the ratio of the number of green chips to the number of yellow chips is 1 : 2. In the second pile, the ratio of the number of green chips to the number of yellow chips is 3 : 5. If Thelma has a total of 20 green chips, then determine the possibilities for the total number of yellow chips.

Themes Algebra, Number Sense

Problem of the Week
Problem C and Solution
Thelma's Chips

Problem

Thelma has two piles of bingo chips. In each pile there are green and yellow chips. In one pile, the ratio of the number of green chips to the number of yellow chips is $1:2$. In the second pile, the ratio of the number of green chips to the number of yellow chips is $3:5$. If Thelma has a total of 20 green chips, then determine the possibilities for the total number of yellow chips.

Solution

Solution 1

In this solution, we first look at all possible combinations of green and yellow chips in the second pile. Since the ratio of the number of green chips to the number of yellow chips in the second pile is $3:5$, we know that the number of green chips in this second pile must be a positive multiple of 3. We also know that there are at most 20 green chips in this pile. Thus, the only possible values for the number of green chips in the second pile are 3, 6, 9, 12, 15, and 18. Then, using the fact that the ratio of the number of green chips to the number of yellow chips is $3:5$, we can determine the number of yellow chips in the second pile for each case. We can also determine the number of green chips in the first pile by subtracting the number of green chips in the second pile from 20. Finally, we can determine the number of yellow chips in the first pile by multiplying the number of green chips in the first pile by 2. This information for each case is summarized in the table below.

Number of green chips in pile 2	Number of yellow chips in pile 2	Number of green chips in pile 1	Number of yellow chips in pile 1	Total number of yellow chips
3	5	$20-3=17$	34	$5+35=39$
6	10	$20-6=14$	28	$10+28=38$
9	15	$20-9=11$	22	$15+22=37$
12	20	$20-12=8$	16	$20+16=36$
15	25	$20-15=5$	10	$25+10=35$
18	30	$20-18=2$	4	$30+4=34$

Therefore, there are six possible values for the total number of yellow chips. There could be 34, 35, 36, 37, 38, or 39 yellow chips in total.

Solution 2

Let a represent the number of green chips in the first pile, where a is a positive integer. Since the ratio of green chips to yellow chips in this pile is $1:2$, then there are $2a$ yellow chips in this pile.

Let $3b$ represent the number of green chips in the second pile, where b is a positive integer. Since the ratio of green chips to yellow chips in this pile is $3:5$, then there are $5b$ yellow chips in this pile.

In total, there are 20 green chips, so $a+3b=20$. Also, the total number of yellow chips is equal to $2a+5b$.

We consider all the possible values for positive integers a and b that satisfy the equation $a+3b=20$. Using these values of a and b, we can then find the possible values of $2a+5b$, and hence the possible values for the total number of yellow chips.

The results are summarized in the table below.

$a+3b$	b	a	$2a+5b$
20	1	17	39
20	2	14	38
20	3	11	37
20	4	8	36
20	5	5	35
20	6	2	34

Therefore, there are six possible values for the total number of yellow chips. There could be 34, 35, 36, 37, 38, or 39 yellow chips in total.

Data Management (D)

TAKE ME TO THE COVER

Problem of the Week
Problem C
Coin Collection

Arya has never travelled to another country, but has a collection of foreign coins given to him by friends and family who have. In his collection he has 10 coins from Africa, 6 coins from Asia, 7 coins from South America, and 8 coins from Europe.

One day Arya's grandfather added some Australian coins to the collection. After he did that, he told Arya that if he took a coin at random from the collection, the probability of it being from either Africa or Asia was $\frac{4}{9}$.

How many Australian coins did Arya's grandfather add to the collection?

Problem of the Week
Problem C and Solution
Coin Collection

Problem

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How many Australian coins did Arya's grandfather add to the collection?

Solution

In order to determine the probability of a randomly selected coin being from either Africa or Asia, we divide the number of coins from Africa or Asia by the total number of coins in the collection. In other words,

$$\text{Probability of selecting a coin from Africa or Asia} = \frac{\text{Number of coins from Africa or Asia}}{\text{Total number of coins}}$$

From here we will present two different solutions to this problem.

Solution 1

When Arya's grandfather adds Australian coins to the collection, this does not change the number of coins from Africa or Asia. Therefore the number of coins from Africa or Asia in the collection is $10 + 6 = 16$. We are also told that the probability of drawing a coin from Africa or Asia is $\frac{4}{9}$. We can substitute these values into our equation.

$$\begin{aligned}\text{Probability of selecting a coin from Africa or Asia} &= \frac{\text{Number of coins from Africa or Asia}}{\text{Total number of coins}}\\ \frac{4}{9} &= \frac{16}{\text{Total number of coins}}\end{aligned}$$

Since $\frac{4}{9} = \frac{16}{36}$, it follows that

$$\frac{16}{36} = \frac{16}{\text{Total number of coins}}$$

Therefore, the total number of coins in the collection is 36.

Originally there were $10 + 6 + 7 + 8 = 31$ coins in the collection, and then Arya's grandfather added some Australian coins. Since there were 36 coins in the collection after the Australian coins were added, it follows that Arya's grandfather must have added $36 - 31 = 5$ Australian coins to the collection.

Solution 2

This solution uses algebra, which may be beyond what some students at this level are familiar with.

Let n represent the number of Australian coins that Arya's grandfather added to the collection. Then the number of coins from Africa or Asia in the collection is $10 + 6 = 16$, and the total number of coins in the collection is $10 + 6 + 7 + 8 + n = 31 + n$. We are also told that the probability of drawing a coin from Africa or Asia is $\frac{4}{9}$. We can substitute these values into our equation.

$$\text{Probability of selecting a coin from Africa or Asia} = \frac{\text{Number of coins from Africa or Asia}}{\text{Total number of coins}}$$
$$\frac{4}{9} = \frac{16}{31+n}$$

Since $4 \times 4 = 16$, it follows that $9 \times 4 = 31 + n$. We can simplify and solve this equation to find the value of n.

$$\begin{aligned} 9 \times 4 &= 31 + n \\ 36 &= 31 + n \\ 36 - 31 &= n \\ 5 &= n \end{aligned}$$

Therefore, Arya's grandfather added 5 Australian coins to the collection.

Problem of the Week
Problem C
Fair Game?

For a school mathematics project, Zesiro and Magomu created a game that uses two special decks of six cards each. The cards in one deck are labelled with the even numbers 2, 4, 6, 8, 10, and 12, and the cards in the other deck are labelled with the odd numbers 1, 3, 5, 7, 9, and 11.

A turn consists of Zesiro randomly choosing a card from the deck with even-numbered labels and Magumo randomly choosing a card from the deck with odd-numbered labels. These two cards make a pair of cards. After a pair of cards is chosen, they perform the following steps.

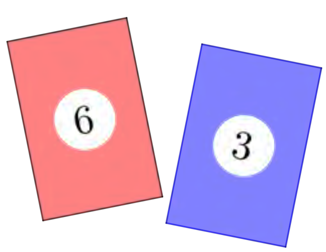
1. They determine the sum, S, of the numbers on the cards. For example, if Zesiro chooses the card labelled with a 6 and Magumo chooses the card labelled with a 3, then $S = 6 + 3 = 9$.

2. Using S, they determine, D, the digit sum. If S is a single digit number, then D is equal to S. If S is a two-digit number, then D is the sum of the two digits of S. For example, if Zesiro chooses the card labelled with a 6 and Magumo chooses the card labelled with a 3, then the sum and the digit sum are both 9. If Zesiro chooses the card labelled with a 10 and Magumo chooses the card labelled with a 5, then the sum is $S = 10 + 5 = 15$ and the digit sum is $D = 1 + 5 = 6$. If Zesiro chooses the card labelled with a 10 and Magumo chooses the card labelled with a 9, then the sum is $S = 10 + 9 = 19$ and the digit sum is $D = 1 + 9 = 10$.

Zesiro gets a point if the digit sum, D, is a multiple of 4.

Magomu gets a point if the number on one of the cards is a multiple of the number on the other card.

Is this game fair? That is, do Zesiro and Magomu have the same probability of getting a point on any turn? Justify your answer.



Theme Data Management

Problem of the Week
Problem C and Solution
Fair Game?

Problem

For a school mathematics project, Zesiro and Magomu created a game that uses two special decks of six cards each. The cards in one deck are labelled with the even numbers 2, 4, 6, 8, 10, and 12, and the cards in the other deck are labelled with the odd numbers 1, 3, 5, 7, 9, and 11.

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Magomu gets a point if the number on one of the cards is a multiple of the number on the other card.

Is this game fair? That is, do Zesiro and Magomu have the same probability of getting a point on any turn? Justify your answer.

Solution

To solve this problem, we will create a table where the columns show the possible choices for the even-numbered card, the rows show the possible choices for the odd-numbered card, and each cell in the body of the table gives the sum of the corresponding pair of cards.

		Even Card					
		2	4	6	8	10	12
Odd Card	1	3	5	7	9	11	13
	3	5	7	9	11	13	15
	5	7	9	11	13	15	17
	7	9	11	13	15	17	19
	9	11	13	15	17	19	21
	11	13	15	17	19	21	23

From the table, we see that the total number of possible pairs is $6 \times 6 = 36$.

We create another table where the columns show the possible choices for the even-numbered card, the rows show the possible choices for the odd-numbered card, and each cell in the body of the table gives the digit sum of the corresponding pair of cards.

		Even Card					
		2	4	6	8	10	12
Odd Card	1	3	5	7	9	$1+1=2$	$1+3=4$
	3	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$
	5	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$
	7	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$
	9	$1+1=2$	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$
	11	$1+3=4$	$1+5=6$	$1+7=8$	$1+9=10$	$2+1=3$	$2+3=5$

If the digit sum is a multiple of 4, then Zesiro gets a point. In the table there are two digit sums, 4 and 8, that are multiples of 4. The digit sum 4 occurs six times in the table and the digit sum 8 occurs four times in the table. This totals ten possible outcomes for Zesiro, and so his probability of scoring a point on any pair is $\frac{10}{36}$.

Magomu has far less work to determine when he gets a point. None of the odd numbers are multiples of the even numbers. All multiples of even numbers are even and hence will never be odd.

Whenever a 1 is chosen, Magomu will score a point. That is, each of the six even numbers is a multiple of 1.

When a 3 is chosen, Magomu will score a point if the number on the face of the even-numbered card is a 6 or 12. That is, only two of the even numbers are multiples of 3.

When a 5 is chosen, Magomu will score a point if the number on the face of the even-numbered card is a 10. That is, only one of the even numbers is a multiple of 5.

None of the numbers in the deck containing only even numbers is a multiple of 7, 9, or 11.

So Magomu will score a point on $6 + 2 + 1 = 9$ of the 36 possible pairs. Therefore, Magomu's probability of scoring a point on any pair is $\frac{9}{36}$.

The game is not fair since Zesiro's probability of scoring a point on any pair is greater than Magomu's probability of scoring a point on any pair.

Problem of the Week

Problem C

Same Same

The mean (average), the median, and the only mode of the five numbers 15, 12, 14, 19, and n are all equal. Determine the value of n.

Theme Data Management

Problem of the Week

Problem C and Solution

Same Same

Problem

The mean (average), the median, and the only mode of the five numbers $15,\ 12,\ 14,\ 19$, and n are all equal. Determine the value of n.

Solution

For the five numbers $15,\ 12,\ 14,\ 19$, and n to have a single mode, n must equal one of the existing numbers in the list: $15,\ 12,\ 14$, or 19. It follows that the mean (average), median, and mode must all equal n.

Since there five numbers, and five is an odd number, the median will be equal to the number in the middle position when the five numbers are written in increasing order. When we write the existing numbers in increasing order, we obtain $12,\ 14,\ 15,\ 19$. Since n is equal to the median, and must also equal one of the existing numbers, the only possibilities are $n = 14$ or $n = 15$.

If $n = 14$, then the mean of the five numbers is $\dfrac{12+14+14+15+19}{5} = 14.8$, which is not equal to 14.

If $n = 15$, then the mean of the five numbers is $\dfrac{12+14+15+15+19}{5} = 15$.

Then the mean, median, and mode are all equal to 15. Therefore, the value of n is 15.

Computational Thinking (C)

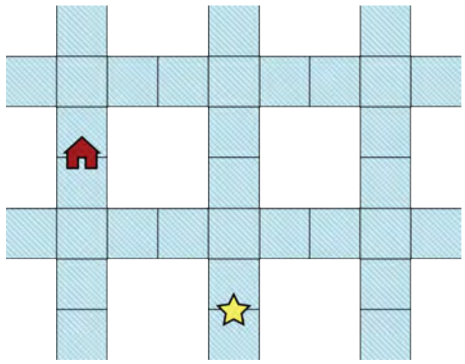
TAKE ME TO THE COVER

Problem of the Week

Problem C

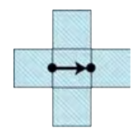
Crossing Canals

Koji is rowing his boat on a busy canal system near his home. The following diagram shows the canal system with a star representing Koji's current location and the house representing the location of his home.

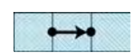


From Koji's extensive canal experience, he knows the following:

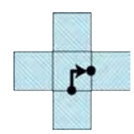
1. Rowing straight across an intersection square takes 30 seconds.



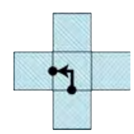
2. Rowing straight across a square that is not an intersection takes 20 seconds.



3. Turning right at an intersection takes 15 seconds.



4. Turning left at an intersection takes 270 seconds, due to heavy traffic.



5. It is not possible make U-turns or reverse direction.

Calculate the shortest amount of time it will take Koji to row home from his current position, using only the canals shown.

Not printing this page? You can use our interactive worksheet.

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

THEME COMPUTATIONAL THINKING

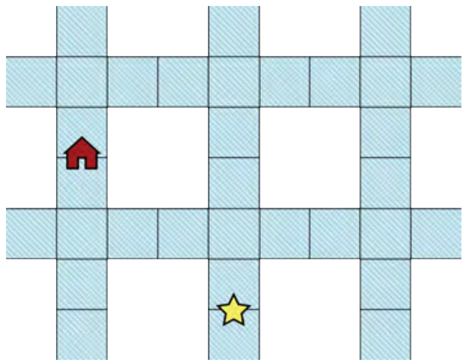
Problem of the Week

Problem C and Solution

Crossing Canals

Problem

Koji is rowing his boat on a busy canal system near his home. The following diagram shows the canal system with a star representing Koji's current location and the house representing the location of his home.



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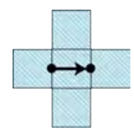
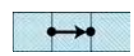
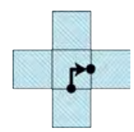


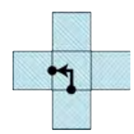
2. Rowing straight across a square that is not an intersection takes 20 seconds.



3. Turning right at an intersection takes 15 seconds.



4. Turning left at an intersection takes 270 seconds, due to heavy traffic.



5. It is not possible make U-turns or reverse direction.

Calculate the shortest amount of time it will take Koji to row home from his current position, using only the canals shown.

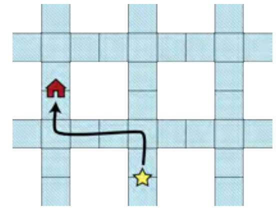
Not printing this page? You can use our interactive worksheet.

Solution

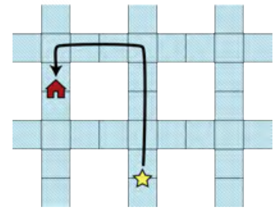
Let R represent a right turn, L represent a left turn, X represent a move straight across an intersection square, and N represent a move straight across a non-intersection square.

We will now consider different routes and calculate the rowing time for each.

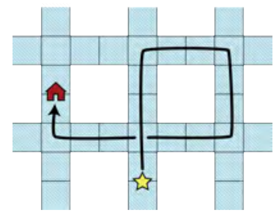
- The shortest route by distance is shown. This corresponds to $N \to L \to N \to N \to R \to N$. Using the given times it would take $4 \times 20 + 1 \times 270 + 1 \times 15 = 365$ seconds.



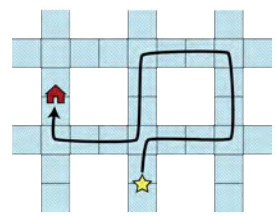
- A second route is shown. This corresponds to $N \to X \to N \to N \to L \to N \to N \to L \to N$. Using the given times it would take $6 \times 20 + 2 \times 270 + 1 \times 30 = 690$ seconds. This route is longer than the first and takes much more time.



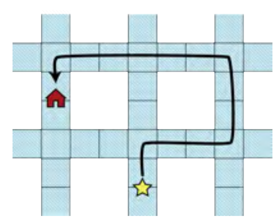
- A third route is shown. This corresponds to $N \to X \to N \to N \to R \to N \to N \to R \to N \to N \to R \to N \to N \to X \to N \to N \to R \to N$. Using the given times it would take $12 \times 20 + 2 \times 30 + 4 \times 15 = 360$ seconds. This route is longer than the previous two, but takes the least amount of time, so far.



- A fourth route is shown. This corresponds to $N \to R \to N \to N \to L \to N \to N \to L \to N \to N \to L \to N \to N \to R \to N \to N \to R \to N$. Using the given times it would take $12 \times 20 + 3 \times 15 + 3 \times 270 = 1095$ seconds. This route takes much longer than the previous routes, due to all the left turns.



- A fifth route is shown. This corresponds to $N \to R \to N \to N \to L \to N \to N \to L \to N \to N \to X \to N \to N \to L \to N$. Using the given times it would take $10 \times 20 + 1 \times 15 + 3 \times 270 + 1 \times 30 = 1055$ seconds. This route also takes more time than the third route.

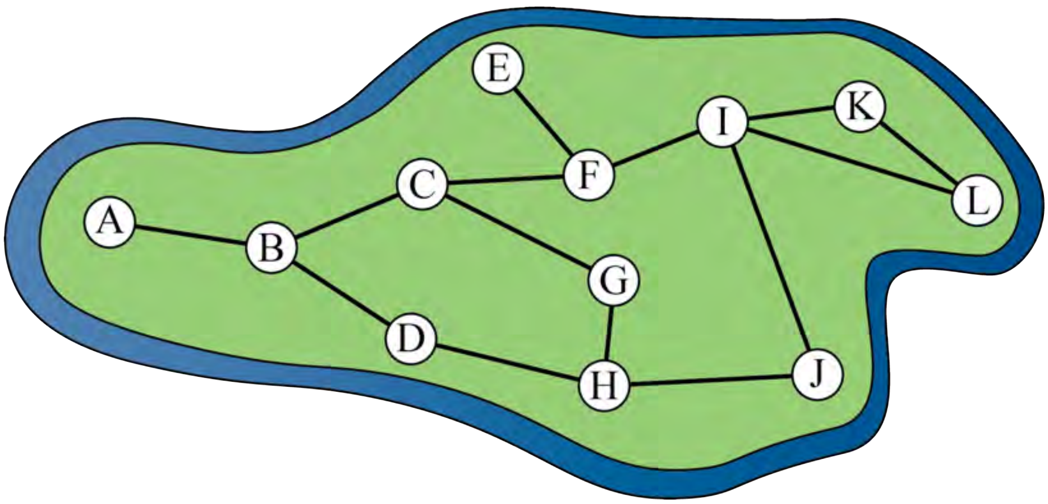


There are other routes that could be checked out but they include at least one of the above routes, so would not be the fastest.

Therefore, the shortest amount of time it will take Koji to row home is 360 seconds.

Problem of the Week
Problem C
Building Community

An island contains twelve small towns that are connected by roads as shown in the diagram below. The towns are labelled with the letters A through L and the roads connecting the towns are indicated by line segments.



The mayor of the island is going to build community centres in some of the towns so that each town either has its own community centre, or is connected by a single road to a town that has a community centre.

(a) If the mayor chooses to build five community centres, which five towns could the mayor choose?

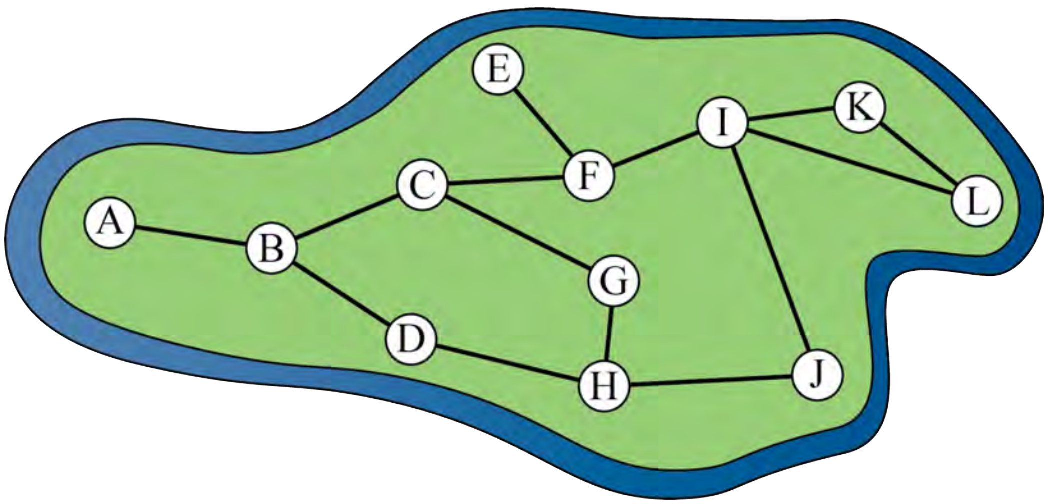
(b) What is the fewest number of community centres the mayor needs to build?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

Problem of the Week
Problem C and Solution
Building Community

Problem

An island contains twelve small towns that are connected by roads as shown in the diagram below. The towns are labelled with the letters A through L and the roads connecting the towns are indicated by line segments.



The mayor of the island is going to build community centres in some of the towns so that each town either has its own community centre, or is connected by a single road to a town that has a community centre.

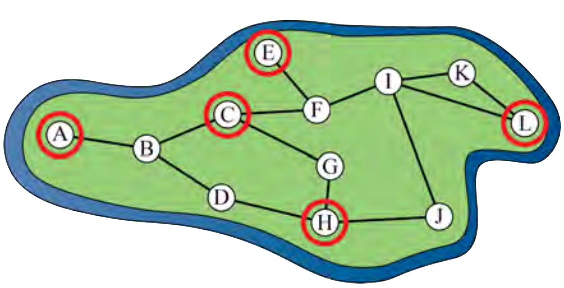
(a) If the mayor chooses to build five community centres, which five towns could the mayor choose?

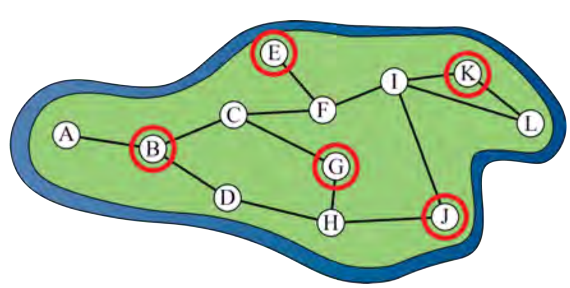
(b) What is the fewest number of community centres the mayor needs to build?

This problem was inspired by a past Beaver Computing Challenge (BCC) problem.

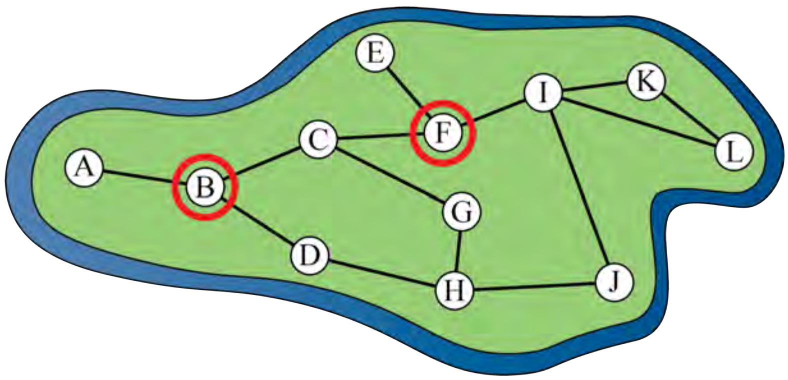
Solution

(a) There are many possible answers. Two possibilities are shown below, where the chosen towns are circled.

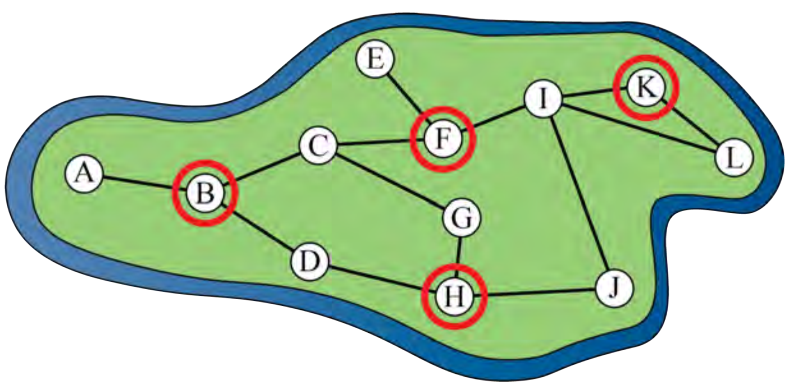




(b) There are two towns that have only one road leading out of them, namely towns A and E. Since each town must either have its own community centre or be connected by a single road to a town that has a community centre, we need to put one community centre in town A or B, and another community centre in town E or F. Since towns B and F are connected to other towns as well, they are the better choices if we want to build the fewest number of community centres.



After choosing towns B and F, the remaining towns that don't have their own community centre and are not connected by a single road to a town that has a community centre are towns G, H, J, K, and L. None of these five towns are connected to all the others, and there is no town that is connected to all five of the towns. So we need at least two more community centres to cover the remaining five towns. Choosing towns K and H would work.



Therefore, the fewest number of community centres the mayor needs to build is four.

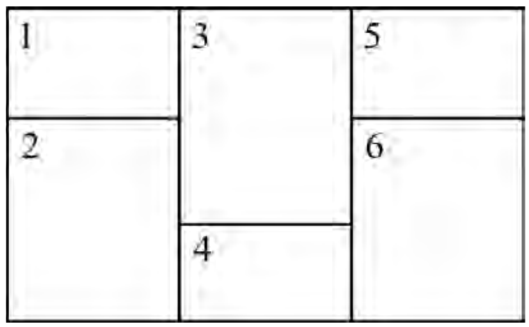
Note that this is not the only group of four towns that we could have chosen. There are several other possibilities.

Problem of the Week

Problem C

Adding Some Colour 1

Martina and Zahra play a game where they take turns shading regions in the diagram shown.



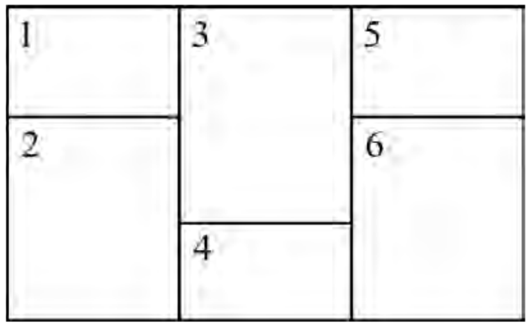
On their turn, each player shades a region in the diagram that is not bordering another shaded region. After some number of turns, it won't be possible to shade any more regions, and the game will be over. The winner is the player who shaded the last region.

Suppose Martina is the first player to shade a region. Two of the six regions are such that if she shades one of them on her first turn, then she can guarantee that she wins the game, regardless of what Zahra does on her turns. Which two regions are they?

Problem of the Week
Problem C and Solution
Adding Some Colour 1

Problem

Martina and Zahra play a game where they take turns shading regions in the diagram shown.



On their turn, each player shades a region in the diagram that is not bordering another shaded region. After some number of turns, it won't be possible to shade any more regions, and the game will be over. The winner is the player who shaded the last region.

Suppose Martina is the first player to shade a region. Two of the six regions are such that if she shades one of them on her first turn, then she can guarantee that she wins the game, regardless of what Zahra does on her turns. Which two regions are they?

Solution

Shading region 3 or region 4 first will guarantee that Martina wins the game. First we will show why this is true, and then we will show why shading any of the other regions first will not guarantee a win for Martina.

Notice that every region in the diagram borders region 3. So if Martina starts by shading region 3 then it will not be possible to shade any other region and the game will be over. Zahra will not even have a chance to take a turn and Martina will win the game.

Now, if Martina starts by shading region 4, then since regions 2, 3, and 6 border region 4, Zahra will not be able to shade these regions. The only regions that Zahra will be able to shade are regions 1 or 5. Since these regions are not bordering each other, Zahra will shade one of these regions, then Martina will shade the other region and win the game.

Thus, we have shown that shading either region 3 or region 4 first will guarantee a win for Martina.

If Martina started by shading region 1, then Zahra would not be able to shade regions 2 or 3, so she would be left to choose between shading region 4, 5, or 6.

Since region 6 borders regions 4 and 5, if Zahra shaded region 6 then Martina would not be able to shade any regions and Zahra would win the game. Thus, shading region 1 first does not guarantee a win for Martina.

Similarly, if Martina started by shading region 5, then Zahra would not be able to shade regions 3 or 6, so she would be left to choose between shading region 1, 2, or 4. Since region 2 borders regions 1 and 4, if Zahra shaded region 2 then Martina would not be able to shade any regions and Zahra would win the game. Thus, shading region 5 first does not guarantee a win for Martina.

If Martina started by shading region 2, then Zahra would not be able to shade regions 1, 3, or 4, so she would be left to choose between shading region 5 or 6. Since regions 5 and 6 border each other, if Zahra shaded either one of them it would not be possible for Martina to shade any regions and Zahra would win the game. Thus, shading region 2 first does not guarantee a win for Martina.

Similarly, if Martina started by shading region 6, then Zahra would not be able to shade regions 3, 4, or 5, so she would be left to choose between shading region 1 or 2. Since regions 1 and 2 border each other, if Zahra shaded either one of them it would not be possible for Martina to shade any regions and Zahra would win the game. Thus, shading region 6 first does not guarantee a win for Martina.

Therefore, regions 3 and 4 are the only regions that Martina can shade first in order to guarantee that she wins the game, regardless of what Zahra does on her turns.