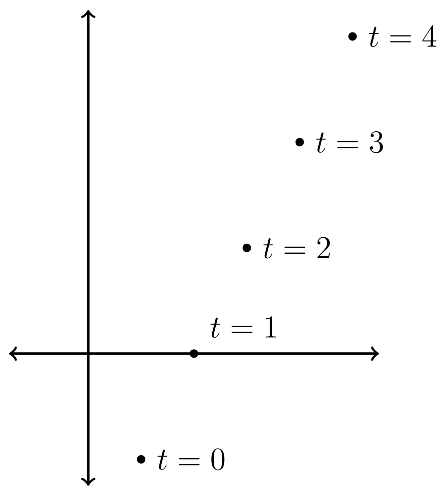
Problem of the Month

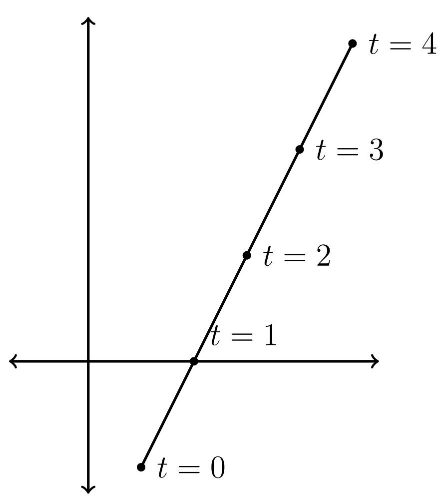
Solution to Problem 7: April 2024

(a) (i) When $t = 0$, the position is $(1 + 0, -2 + 2(0)) = (1, -2)$. When $t = 1$, the position is $(1+1, -2+2(1)) = (2, 0)$. When $t = 2$, the position is $(3, 2)$, when $t = 3$, the position is $(4, 4)$, and when $t = 4$, the position is $(5, 6)$. The plot of these positions is below



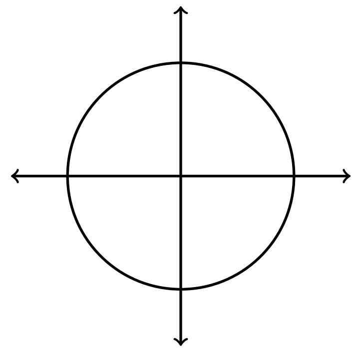
(ii) For each t, we have $x = 1 + t$ and $y = -2 + 2t$. Solving for t in the equation $x = 1 + t$ gives $t = x - 1$, which can be substituted into the second equation to get $y = -2 + 2(x - 1) = -2 + 2x - 2 = 2x - 4$. Therefore, every point of the form $(x, y) = (t + 1, -2 + 2t)$ satisfies $y = 2x - 4$.

(iii) From part (ii), the points that the particle occupies are all on the line with equations $y = 2x - 4$. We care about the points on this line from $t = 0$ to $t = 4$ inclusive, so the plot is just the line segment connecting $(1, -2)$ (the point for $t = 0$) to $(5, 6)$ (the point for $t = 4$). The plot is below.



(b) We have $x = \cos t$ and $y = \sin t$, so for any position (x, y) that the particle occupies, $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$. Therefore, the particle is always somewhere on the unit circle.

Indeed, every point on the unit circle has coordinates $(\cos t, \sin t)$ for exactly one real number t with $0 \leq t < 2\pi$. The graph of the path of the particle is



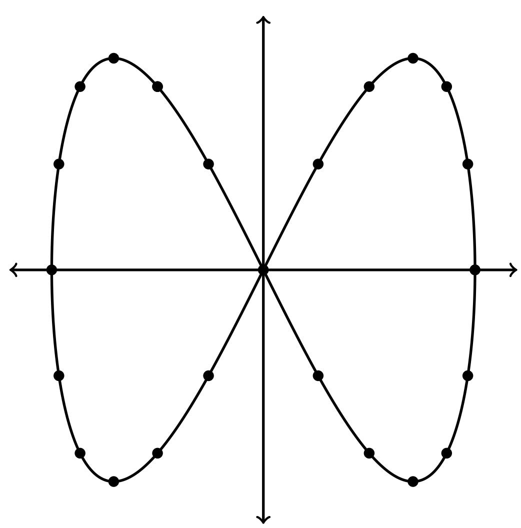
(c) (i) In each of the three tables below, the left column contains values of t, the middle column contains the corresponding values of $\cos t$, and the right column contains the corresponding values of $\sin 2t$. Although it is not particularly difficult to write down exact values for each of the trigonometric ratios in the table, we want to plot the points so the values are all given rounded to three digits past the decimal point.

t	$\cos t$	$\sin 2t$
0	1	0
$\frac{\pi}{12}$	0.966	0.5
$\frac{2\pi}{12}$	0.866	0.866
$\frac{3\pi}{12}$	0.707	1
$\frac{4\pi}{12}$	0.5	0.866
$\frac{5\pi}{12}$	0.259	0.5
$\frac{6\pi}{12}$	0	0
$\frac{7\pi}{12}$	-0.259	-0.5

t	$\cos t$	$\sin 2t$
$\frac{8\pi}{12}$	-0.5	0.866
$\frac{9\pi}{12}$	-0.7071	-1
$\frac{10\pi}{12}$	-0.866	-0.866
$\frac{11\pi}{12}$	-0.966	-0.5
π	-1	0
$\frac{13\pi}{12}$	-0.966	0.5
$\frac{14\pi}{12}$	-0.866	0.866
$\frac{15\pi}{12}$	-0.707	1

t	$\cos t$	$\sin 2t$
$\frac{16\pi}{12}$	-0.5	0.866
$\frac{17\pi}{12}$	-0.259	0.5
$\frac{18\pi}{12}$	0	0
$\frac{19\pi}{12}$	0.259	-0.5
$\frac{20\pi}{12}$	0.5	-0.866
$\frac{21\pi}{12}$	0.707	-1
$\frac{22\pi}{12}$	0.866	-0.866
$\frac{23\pi}{12}$	0.966	-0.5
$\frac{24\pi}{12}$	1	0

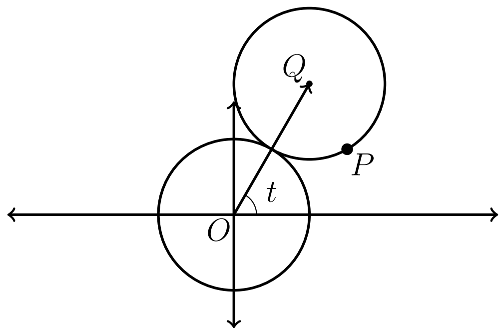
Below is a plot of the points above including a sketch of the curve.



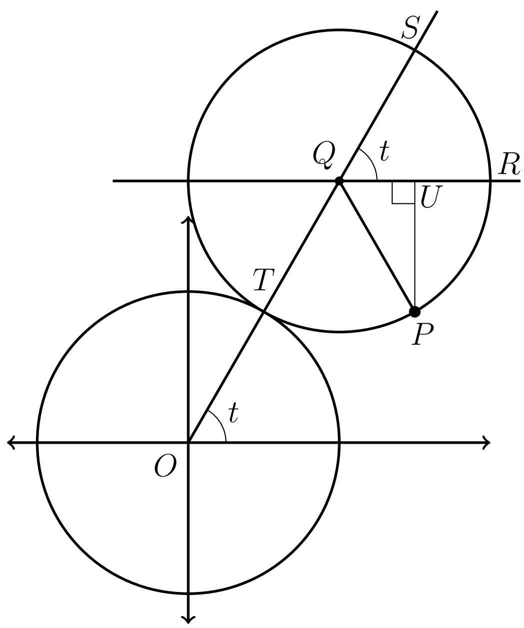
(ii) Every point (x, y) on the parametric curve satisfies $x = \cos t$ and $y = \sin 2t$ for some real number t. Using the double-angle formula for $\sin 2t$, we get $y = 2 \sin t \cos t$, so $y^2 = 4\sin^2 t\cos^2 t$. By the Pythagorean identity, $1 - x^2 = 1 - \cos^2 t = \sin^2 t$. Substituting gives $y^2 = 4(1-x^2)x^2 = 4x^2 = 4x^4$.

(d) We will label the point on Circle 2 that is originally at $(1, 0)$ by P and we will label the centre of Circle 2 by Q. Since the circumferences of the circles are the same, Circle 2 will return to its original position after rolling exactly once around Circle 1. This means Q will travel exactly once around the circle of radius $1 + 1 = 2$ centred at the origin.

We will let t represent the angle made by the positive x-axis and the ray from the origin to Q. For example, the diagram below depicts the position of the outer circle when $t = \frac{\pi}{3}$. The origin is labelled by O.



We wish to find the coordinates of P in terms of the angle t. In the diagram below, we have added to the previous diagram a line through the origin and Q, a line segment connecting Q to P, as well as a horizontal line through Q. The horizontal line intersects Circle 2 at R to the right of Q. The line through O and Q intersects the point of tangency of the two circles at T and it also intersects Circle 2 at S (the points S and T are different). A perpendicular has also been drawn from P to QR intersecting QR at U.



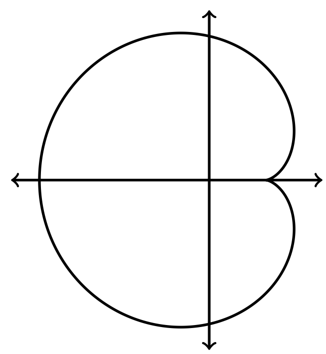
The line connecting the centres of tangent circles always passes through the point of tangency, which justifies the implicit claim in the previous paragraph that OQ passes through the point of tangency. It also implies that the length of OQ is $1 + 1 = 2$. Therefore, the coordinates of Q are $(2\cos t, 2\sin t)$. Since QR is horizontal by construction, QR is parallel to the x axis, which implies $\angle SQR = t$.

Because Circle 2 is rolling along Circle 2 without slipping, the arc from T to the x axis along Circle 1 has the same length as the arc TP. This means $\angle TQP = t$ as well since the two circles have the same radius. Since TQS is a line, we get that $\angle RQP = \pi - 2t$.

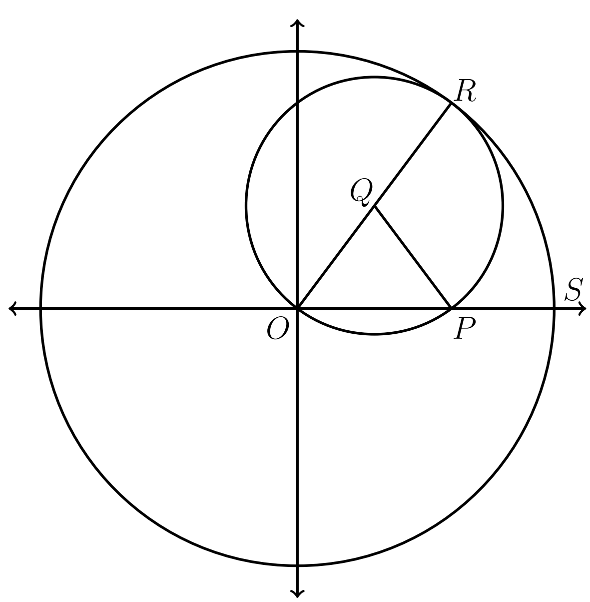
Suppose Q has coordinates (q_1, q_2) and P has coordinates (p_1, p_2). The radius of Circle 2 is 1, so $QP = 1$. This means $p_1 - q_1 = \cos(\pi - 2t)$ and $q_2 - p_2 = \sin(\pi - 2t)$. Thus,

$$\begin{aligned}(p_1, p_2) &= \big(q_1 + (p_1 - q_1), q_2 - (q_2 - p_1)\big) \\ &= \big(2\cos t + \cos(\pi - 2t), 2\sin t - \sin(\pi - 2t)\big) \\ &= (2\cos t - \cos 2t, 2\sin t - \sin 2t)\end{aligned}$$

where the last equality is by trigonometric identities. Therefore, $x(t) = 2\cos t - \cos 2t$ and $y(t) = 2\sin t - \sin 2t$. A plot of this curve is given below.



(e) As Circle 2 rolls around the inside of Circle 1, if a line is drawn through the point of tangency perpendicular to the mutual tangent, then the line will pass through the centre of both circles. Such a line contains a diameter of both circles, and since the radius of Circle 1 equals the diameter of Circle 2, the centre of Circle 1 is always on Circle 2. In the diagram below, Circle 2 has been rotated by some positive angle between 0 and $\frac{\pi}{2}$. The origin is labelled by O, the current point of tangency is labelled by R, the centre of Circle 2 is labelled by Q, the other point at which Circle 2 intersects the x-axis is labelled by P, and $(2, 0)$ is labelled by S.

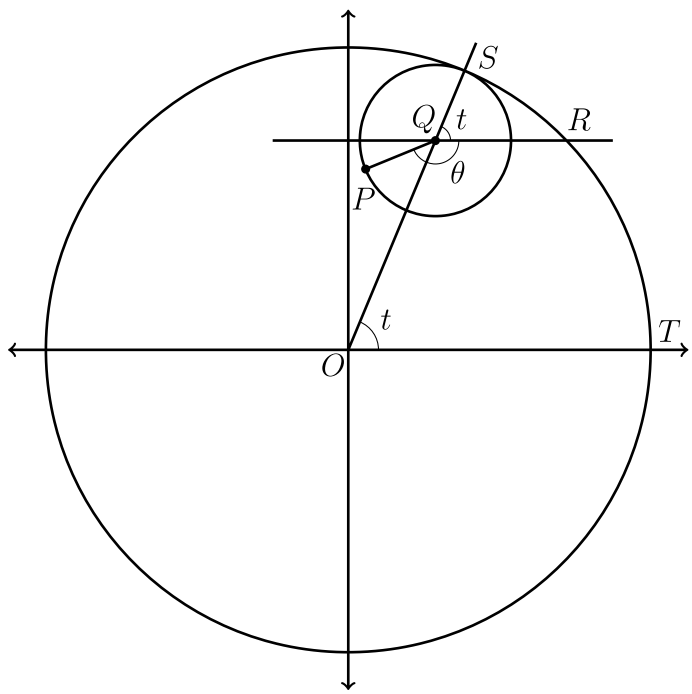


We wish to determine the current location of the point on Circle 2 that started at S. Circle 2 does not slip as it rolls, so this point is the same distance from R in the clockwise direction along both circles.

Since O is on the circumference of Circle 2 and Q is the centre of Circle 2, a well-known circle property implies that $\angle PQR = 2\angle POR$. The radius of Circle 1 is 2, so provided we measure angles in radians, the length of the arc PS is $2\angle POR$. The radius of Circle 2 is 1, so the length of arc PR is $\angle PQR$. These quantities are equal, so the arcs PR and PS have equal length. Therefore, the original point of tangency is at P.

By drawing similar diagrams for obtuse and reflex angles, it can be similarly shown that the original point of tangency is always on the diameter of Circle 1. Now note half the circumference of Circle 1 is equal in length to the circumference of Circle 2, so when Circle 2 has rolled exactly half way around Circle 1, the original point of tangency is exactly where Circle 1 intersects the negative x-axis. It follows by symmetry that the original point of tangency will go from $(2, 0)$ to $(-2, 0)$ and back to $(2, 0)$ as Circle 2 rolls around Circle 1.

(f) We will first work out, in general, a pair of parametric equations for P, the original point of tangency. As in part (d), O is the origin, Q is the centre of Circle 2, and the measure of the angle made by the positive x-axis and the ray from O to Q is t. In the diagram below, a horizontal line is drawn through Q intersecting Circle 2 at R to the right of Q and the line defined by OQ, for the same reasoning that was used in part (d), passes through the mutual point of tangency labelled by S. We label $(1, 0)$ by T.



As mentioned, O, Q, and S are on a line, and so $OQ = OS - QS = 1 - r$. Therefore, the coordinates of Q are $\big((1-r)\cos t, (1-r)\sin t\big)$. If we let θ be $\angle PQR$, then by reasoning similar to that which was used in part (d), the coordinates of P are

$$((1-r)\cos t + r\cos\theta, (1-r)\sin t - r\sin\theta)$$

For now, assume that t is small enough that Circle 2 has not yet rolled far enough for P to have returned to the circumference of Circle 1. The length of arc SP on Circle 2 is equal to the length of arc ST on Circle 1, which is equal to t since the radius of Circle 1 is 1 (provided we use radians as our unit of angle measure). Because QR is parallel to OT, $\angle SQR = \angle SOT = t$, so $\angle SQP = t + \theta$ (note that this angle is measured clockwise from S, so in the diagram, the angle measures more than π radians). The radius of Circle 2 is r, so the length of arc SP is $r(t+\theta)$. The arcs ST and SP are equal, so $t = rt + r\theta$, which can be solved for θ to get $\theta = \dfrac{t - tr}{r}$, and we note that $0 < r < 1$, so $t - tr$ is positive.

Now suppose that t is such that P has already returned to Circle 1 k times. Between consecutive times that P returns to Circle 1, the arc ST increases in length by $2\pi r$, the circumference of Circle 2. Therefore, instead of the equation $t = rt + r\theta$, we get $t = rt + r\theta + 2\pi rk$ since the arc length from S to T is still t, but to get equality, we need to account for the k complete revolutions of Circle 2.

Solving this equation for θ gives $\theta = \dfrac{t - rt - 2\pi rk}{r} = \dfrac{t - rt}{r} - 2\pi k$. However, since we will be taking sin and cos of θ, we can ignore the quantity $2\pi k$ since k is an integer. Therefore, the coordinates of P when Q makes an angle of t with the positive x-axis are

$$\left((1-r)\cos t + r\cos\left(\frac{t-tr}{r}\right), (1-r)\sin t - r\sin\left(\frac{t-tr}{r}\right)\right)$$

Before answering the given questions, we note that if $r = \dfrac{1}{2}$, (that is, the radius of Circle 2 is twice that of Circle 1), then the coordinates simplify to $(\cos t, 0)$, meaning the point P always has a y-coordinate of 0. This gives another proof of the result in (e).

(i) When $r = \dfrac{1}{4}$, the coordinates of P are

$$\left(\frac{3}{4}\cos t + \frac{1}{4}\cos(3t), \frac{3}{4}\sin t - \frac{1}{4}\sin(3t)\right)$$

Using standard trigonometric identities, one can show that $\cos(3t) = 4\cos^3 t - 3\cos t$ and $\sin(3t) = 3\sin t - 4\sin^3 t$. Substituting into the coordinates above, we get

$$\begin{aligned} x(t) &= \frac{1}{4}(3\cos t + \cos(3t)) \\ &= \frac{1}{4}(3\cos t + 4\cos^3 t - 3\cos t) \\ &= \cos^3 t \end{aligned} \qquad \begin{aligned} y(t) &= \frac{1}{4}(3\sin t - \sin(3t)) \\ &= \frac{1}{4}(3\sin t - 3\sin t + 4\sin^3 t) \\ &= \sin^3 t \end{aligned}$$

leading to the rather tidy expression for the coordinates of $(\cos^3 t, \sin^3 t)$. Therefore, $\sqrt[3]{x} = \cos t$ and $\sqrt[3]{y} = \sin t$, so $(\sqrt[3]{x})^2 + (\sqrt[3]{y})^2 = \cos^2 t + \sin^2 t = 1$.

(ii) For $r = \dfrac{1}{3}$, we get

$$\begin{aligned} x(t) &= \frac{2}{3}\cos t + \frac{1}{3}\cos(2t) \\ &= \frac{1}{3}(2\cos t + \cos 2t) \end{aligned} \qquad \begin{aligned} y(t) &= \frac{2}{3}\sin t - \frac{1}{3}\sin(2t) \\ &= \frac{1}{3}(2\sin t - \sin 2t) \end{aligned}$$

For $r = \dfrac{2}{3}$, we get

$$x(t) = \frac{1}{3}\cos t + \frac{2}{3}\cos\left(\frac{t}{2}\right) \qquad\qquad y(t) = \frac{1}{3}\sin t - \frac{2}{3}\sin\left(\frac{t}{2}\right)$$
$$= \frac{1}{3}\left(\cos t + 2\cos\frac{t}{2}\right) \qquad\qquad = \frac{1}{3}\left(\sin t - 2\sin\frac{t}{2}\right)$$

Focusing for a moment on the equations for $r = \dfrac{2}{3}$, if we substitute $t = 2\pi$, we get $x(2\pi) = -\dfrac{1}{3}$ and $y(2\pi) = 0$, which means the point initially at $(1,0)$ on Circle 2 has not returned to its original position yet. This is because the circumference of Circle 1 is not an integer multiple of the circumference of Circle 2. Indeed, the circumference of Circle 1 is 2π, but the circumference of Circle 2 is $\dfrac{4\pi}{3}$.

When Circle 2 first reaches the point where it is tangent to Circle 1 at $(1,0)$, some points on Circle 2 have been tangent to Circle 1 more than once. To be precise, $2\pi - \dfrac{4\pi}{3} = \dfrac{2\pi}{3}$, so every point on Circle 2 has been tangent to Circle 1 at least once, but the points on an arc of length $\dfrac{2\pi}{3}$ on Circle 2 have been tangent to Circle 1 twice. Specifically, since $2 \times \dfrac{2\pi}{3} = \dfrac{4\pi}{3}$, exactly half of the points on Circle 2 have been tangent to Circle 1 twice and the rest have been tangent once.

If Circle 2 rolls around Circle 1 again, then half the points on Circle 2 will be tangent to Circle 1 exactly twice, and the other half will be tangent exactly once. This gives a total of three times for each point. Indeed, $3\dfrac{4\pi}{3} = 4\pi = 2(2\pi)$, so three times the circumference of Circle 2 is an integer multiple of the circumference of Circle 1. Therefore, P returns to $(1,0)$ for the first time when the circumference of Circle 2 wraps around the inside of the circumference of Circle 1 exactly 2 times.

Algebraically, if we substitute $t = 4\pi$, we get $x(4\pi) = 1$ and $y(4\pi) = 0$, so the curve for $r = \dfrac{2}{3}$ is travelled periodically every *two* times Circle 2 rolls along Circle 1.

To avoid confusion, we will refer to the circle with $r = \dfrac{1}{3}$ as Circle 3 and the circle with $r = \dfrac{2}{3}$ as Circle 2. The circumference of Circle 1 is an integer multiple of the circumference of Circle 3, so point P reaches $(1,0)$ after Circle 3 rolls around Circle 1 exactly once.

Suppose the centre of Circle 3 revolves around the origin at a rate of 1 radian per second and the centre of Circle 2 revolves around the origin at a rate of 2 radians per second. By assuming this, we will have that both circles take exactly 2π seconds for P to return to its original position for the first time.

If Circle 3 is instead rolled clockwise around Circle 1, then the position of P will be $(x(2\pi - t), y(2\pi - t))$ where $x(t)$ and $y(t)$ are as given for $r = \dfrac{1}{3}$ above. This is because, for example, point P is in the same position after rolling $\dfrac{\pi}{3}$ radians counterclockwise as it would be if it had rolled $2\pi - \dfrac{\pi}{3}$ radians clockwise. Thus, if Circle 3 rotated clockwise instead of counterclockwise, its position at time t is given by $(x(t), y(t))$ where

$$\begin{aligned} x(t) &= \frac{1}{3}(2\cos(2\pi - t) + \cos(2(2\pi - t)) & y(t) &= \frac{1}{3}(2\sin(2\pi - t) - \sin(2(2\pi - t)) \\ &= \frac{1}{3}(2\cos t + \cos(2t)) & &= \frac{1}{3}(-2\sin t + \sin 2t) \\ &= \frac{1}{3}(\cos 2t + 2\cos t) & &= \frac{1}{3}(\sin 2t - 2\sin t) \end{aligned}$$

If Circle 2 is rotated counterclockwise (as before) at 2 radians per second, then at time t the angle is $2t$, and so the coordinates of P are

$$\begin{aligned} x(t) &= \frac{1}{3}\left(\cos 2t + 2\cos\frac{2t}{2}\right) & y(t) &= \frac{1}{3}\left(\sin 2t - 2\sin\frac{2t}{2}\right) \\ &= \frac{1}{3}(\cos 2t + 2\cos t) & &= \frac{1}{3}(\sin 2t - 2\sin t) \end{aligned}$$

which are the exact same coordinates as Circle 3 at time t. Of course, if Circle 3 is rotated clockwise, then it travels the same path as if it were rotated counterclockwise but in the opposite direction. Thus, if both Circle 2 and Circle 3 are rotated counterclockwise, then P travels the same path in opposite directions.