Problem of the Month

Problem 7: April 2024

Curves in the plane are often given as the set of points (x, y) that satisfy some equation in x and y. For example, the set of points (x, y) that satisfy $y = x^2$ is a parabola, the set of points (x, y) that satisfy $y = 3x + 4$ is a line, and the set of points (x, y) that satisfy $x^2 + y^2 = 1$ is the circle of radius 1 centred at the origin.

Another way to express a curve in the plane is using *parametric* equations. With this type of description, we introduce a third variable, t, called the *parameter*, and each of x and y is given as a function of t. This is useful for describing the position of a point that is moving around the plane. For example, imagine that an ant is crawling around the plane. If its x-coordinate at time t is $x = x(t)$ and its y-coordinate at time t is $y(t)$, then its position at time t is $(x(t), y(t))$.

(a) A particle's position at time t is $(x, y) = (1 + t, -2 + 2t)$. That is, its x-coordinate at time t is $1 + t$ and its y-coordinate at time t is $-2 + 2t$.

(i) Plot the position of the particle at $t = 0$, 1, 2, 3, and 4.

(ii) Show that every position the particle occupies is on the line with equation $y = 2x - 4$.

(iii) Sketch the path of the particle from $t = 0$ through $t = 4$.

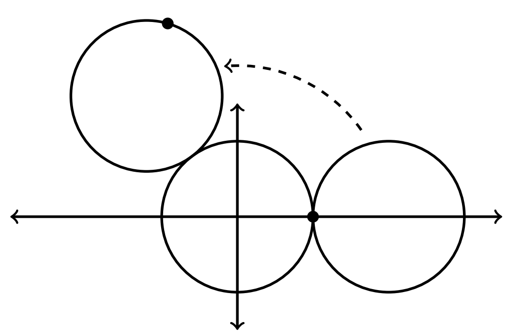
(b) A particle's position at time t is $(\cos(t), \sin(t))$. Sketch the path of the particle from $t = 0$ through $t = 2\pi$.

(c) A particle's position at time t is $(\cos(t), \sin(2t))$.

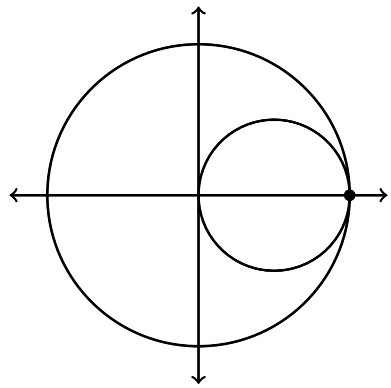
(i) Plot the position of the particle at $t = \dfrac{k\pi}{12}$ for the integers $k = 0$ through $k = 24$. Sketch the path of the particle from $t = 0$ through $t = 2\pi$.

(ii) Show that every position the particle occupies is on the curve with equation $y^2 = 4x^2 - 4x^4$.

(d) Circle 1 is centred at the origin, Circle 2 is centred at $(2, 0)$, and both circles have radius 1. The circles are tangent at $(1, 0)$. Circle 2 is "rolled" in the counterclockwise direction along the outside of the circumference of Circle 1 without slipping. The point on Circle 2 that was originally at $(1, 0)$ (the point of tangency) follows a curve in the plane. Find functions $x(t)$ and $y(t)$ so that the points on this curve are $(x(t), y(t))$ for $0 \leq t \leq 2\pi$.



(e) The setup in this problem is similar to (d). Circle 1 is centred at the origin and has radius 2 and Circle 2 is centred at $(1, 0)$ and has radius 1 so that the two circles are tangent at $(2, 0)$. Circle 2 is rolled around the inside of the circumference of Circle 1 in the counterclockwise direction. Describe the curve in the plane followed by the point on Circle 2 that is initially at $(2, 0)$.



(f) Circle 1 is centred at the origin and has radius 1. Circle 2 has radius $r < 1$, is inside Circle 1, and the two circles are initially tangent at $(1, 0)$. When Circle 2 is rolled around the inside of Circle 1 in the counterclockwise direction, the point on Circle 2 that was initially at $(1, 0)$ will follow some curve in the plane.

(i) Show that when $r = \frac{1}{4}$, the points on the curve satisfy the equation $\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{y}\right)^2 = 1$.

(ii) Show that the curves for $r = \frac{1}{3}$ and $r = \frac{2}{3}$ are exactly the same and that the point initially at $(1, 0)$ travels this curve in opposite directions for the two radii.