Problem of the Month

Problem 5: February 2024

This problem explores a counting technique that uses *binomial coefficients*. **The main problems for this month are on the next page**. For anyone who is not already familiar with the notation of binomial coefficients, the first page includes an introduction with a few standard exercises.

For non-negative integers n and k, the expression $\binom{n}{k}$ denotes the number of ways to choose k objects from n distinct objects. For this reason, the quantity $\binom{n}{k}$ is pronounced "n choose k".

For example, consider the four letters A, B, C, and D. There are 4 ways to select three of them. They are

$$A, B, C \qquad A, B, D \qquad A, C, D \qquad B, C, D$$

Since there are 4 ways to choose 3 of the objects (order does not matter – it only matters which objects are chosen), we have that $\binom{4}{3} = 4$.

Try the following exercises. Hints will be provided, but solutions will not. These exercises are intended as a way to get familiar with the notation, so they may not be explicitly useful in the problems on the next page.

(i) Show that $\binom{5}{2} = 10$.

(ii) What are $\binom{n}{0}$ and $\binom{n}{1}$?

(iii) Convince yourself that $\binom{n}{k} = \binom{n}{n-k}$ for $0 \leq k \leq n$.

(iv) Convince yourself that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ when $0 \leq k < n$.

(v) Convince yourself that $\binom{n}{k}$ is equal to the coefficient of x^k in the expanded form of $(1+x)^n$.

This is why $\binom{n}{k}$ is called a "binomial coefficient".

It turns out that there is a convenient way to compute binomial coefficients using *factorial* notation. If $n \geq k \geq 0$ are integers, then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$. Here, $m!$, pronounced "m factorial", is defined so that $m! = 1$ when $m = 0$ and $m = 1$, and $m! = 1 \times 2 \times 3 \times \cdots \times (m-1) \times m$ for integers $m \geq 2$. It is also a useful exercise to think about why this formula for $\binom{n}{k}$ works.

(a) For a positive integer n, show that $\binom{n+1}{2}$ is the answer to each of these three questions. Think about why the answers to all three questions are the same.

(i) What is $1 + 2 + 3 + \cdots + n$?

(ii) How many pairs (x, y) of non-negative integers satisfy $x + y \leq n - 1$?

(iii) How many triples (x, y, z) of non-negative integers satisfy $x + y + z = n - 1$?

(b) Let $n \geq 0$ and $r \geq 1$ be integers. Show that the following two questions have the same answer and express the common answer as a single binomial coefficient.

(i) How many ways are there to place n indistinguishable balls in r distinguishable cups?

(ii) How many non-negative integer solutions are there to the equation $x_1+x_2+\cdots+x_r = n$?

(c) Let $n \geq 0$ and $r \geq 1$ be integers. By counting r-tuples of non-negative integers $(x_1, x_2, \ldots, x_r)$ that satisfy $x_1 + x_2 + x_3 + \cdots + x_r \leq n$, prove the identity

$$\binom{r-1}{r-1} + \binom{r}{r-1} + \binom{r+1}{r-1} + \binom{r+2}{r-1} + \cdots + \binom{r-1+n}{r-1} = \binom{n+r}{r}$$

Clarification: The quantity on the left in the equation above is the sum of the binomial coefficients $\binom{r-1+m}{r-1}$ where m ranges from 0 to n.

(d) Determine the number of non-negative integers x with $x < 10^{10}$ that have a digit sum of 21

(e) For a positive integer $x = a_n a_{n-1} a_{n-2} \cdots a_1 a_0$ where the a_i are the digits of x, the *alternating sum* of x is the expression $a_0 - a_1 + a_2 - a_3 + \cdots + (-1)^n a_n$. For example, the alternating sum of 744923 is $3 - 2 + 9 - 4 + 4 - 7 = 3$. Determine the number of non-negative integers with at most 8 digits that have an alternating sum equal to 0.

(f) How many ways are there to choose integers a, b, and c such that $1 \leq a < b < c \leq 2024$ and $a + b + c = 2027$?