Problem of the Month
Problem 5: February 2024

Hint

First, some hints on the exercises.

(i) List the two-element subsets of $\{1, 2, 3, 4, 5\}$.

(ii) The answers are 1 and n. It might take a moment of reflection to convince yourself that $\binom{n}{0} = 1$ makes sense.

(iii) If k objects are chosen from a set of n objects, then how many objects are not chosen?

(iv) If you are to choose $k + 1$ integers from the set $\{1, 2, 3, \ldots, n, n + 1\}$, then either $n + 1$ is chosen or it is not.

(v) The quantity $(1 + x)^n$ is equal to the product of n copies of $(1 + x)$. Try expanding $(1 + x)^n$ for a few small values of n without collecting like terms. As an example, think about how an x^3 term could arise during the expansion of $(1 + x)(1 + x)(1 + x)(1 + x)(1 + x)$.

Below are the hints for the main problems.

(a) If you have never seen a proof of (a)(i), try writing the sum $1 + 2 + 3 + \cdots + n$ in reverse order directly under the sum $1+2+3+\cdots+n$. Now add each column. For (a)(ii), consider the possible values of x. For (a)(iii), consider the equation $x + y + z = n - 1$ for a fixed pair (x, y).

(b) Imagine arranging the n identical balls in a row and placing $r - 1$ sticks between them. By doing this, you have partitioned the n balls into r smaller groups.

(c) Introduce a new variable, x_0, and consider the equation $x_0 + x_1 + \cdots + x_r = n$.

(d) The non-negative integers x with $x < 10^{10}$ are exactly the integers that have at most 10 digits. Consider the equation $x_1 + x_2 + \cdots + x_{10} = 21$ where $0 \le x_i \le 9$.

There was an omission in part (d). The original question said "integers" where it should have said "non-negative integers".

(e) This question can be analyzed by examining the equation $x_1 - x_2 + x_3 - x_4 + x_5 - x_6 + x_7 - x_8 = 0$. Rearrange this equation and use the ideas from (b) and (d).

(f) Let $x = a - 1$, $y = b - 1$, and $z = c - 1$. Find the number of non-negative integer solutions to $x + y + z = 2024$ with x, y, and z distinct.