Problem of the Month
Problem 4: January 2024

In this problem, we will enclose a list of positive integers in square brackets. For example, $[1, 2, 4, 7, 9]$ is a list of positive integers of length five. All lists in this problem will be expressed in *non-decreasing* order. To denote the list of consecutive integers from a to b inclusive, we will use the notation $[a : b]$. For example, $[4 : 7]$ denotes the list $[4, 5, 6, 7]$.

Given a list, A, of positive integers, we will define $f(A)$ to be the increasing list of distinct positive integers that are expressible as the sum of some, but not all, of the items in A. A sum is allowed to consist of just one item in A, and each item in A can be used at most once in a particular sum.

For example, if $A = [1, 1, 3, 7]$, then the sums of at least one but not all of the items in A are shown below.

$1 = 1$	$1 + 1 = 2$	$1 + 1 + 3 = 5$
$1 = 1$	$1 + 3 = 4$	$1 + 1 + 7 = 9$
$3 = 3$	$1 + 7 = 8$	$1 + 3 + 7 = 11$
$7 = 7$	$1 + 3 = 4$	$1 + 3 + 7 = 11$
	$1 + 7 = 8$	
	$3 + 7 = 10$	

Therefore, $f(A) = [1, 2, 3, 4, 5, 7, 8, 9, 10, 11]$.

A list, D, of positive integers is called *compressible* if there exists some list A with $f(A) = D$. In this situation, we say that D is compressed by A or that A compresses D. From the example above, we have that $[1, 2, 3, 4, 5, 7, 8, 9, 10, 11]$ is compressible and is compressed by $[1, 1, 3, 7]$.

(a) Find all lists of length four that compress $[1 : 9]$ and explain why no list of length three or less can compress $[1 : 9]$.

(b) Find a list, A, that compresses $[1 : 100]$ and is as short as possible.

(c) Show that for all positive integers n, the list $[1 : n]$ is compressible. For each n, determine the smallest possible length of a list that compresses $[1 : n]$.

(d) Show that for all positive integers $k \geq 3$ there are only finitely many compressible lists of k consecutive positive integers. That is, for each positive integer $k \geq 3$, show that there are only finitely many positive integers m for which $[m : m + k - 1]$ is compressible.

(e) Find the largest positive integer k such that $[5 : k]$ is not compressible. (A full solution will not be provided for this part.)