Problem of the Month
Solution to Problem 3: Three-sign sums

December 2024

1. (a) We have

$$f_3 = \mu(\{1\}) + \mu(\{2\}) + \mu(\{3\}) + \mu(\{1,2\}) + \mu(\{1,3\}) + \mu(\{2,3\}) + \mu(\{1,2,3\})$$
$$= 1 + 2 + 3 + (1+2) + (1+3) + (2+3) + (1+2-3)$$
$$= 18.$$

For f_4, we will organise ourselves a little differently to make computing f_5 a little easier. Notice that if a sublist of $\{1,2,3,4\}$ does not contain 4, then it is a sublist of $\{1,2,3\}$. Therefore we have

$$f_4 = f_3 + \mu(\{4\}) + \mu(\{1,4\}) + \mu(\{2,4\}) + \mu(\{3,4\})$$
$$\quad + \mu(\{1,2,4\}) + \mu(\{1,3,4\}) + \mu(\{2,3,4\}) + \mu(\{1,2,3,4\})$$
$$= 18 + 4 + (1+4) + (2+4) + (3+4)$$
$$\quad + (1+2-4) + (1+3-4) + (2+3-4) + (1+2-3+4)$$
$$= 44.$$

When computing f_4, notice that if we ignore all the 4s that showed up in the sum, what is left over is again a copy of f_3. With this in mind, for f_5 we have

$$f_5 = f_4 + 5 + (1+5) + (2+5) + (3+5) + (4+5)$$
$$\quad + (1+2-5) + (1+3-5) + (1+4-5)$$
$$\quad + (2+3-5) + (2+4-5) + (3+4-5)$$
$$\quad + (1+2-3+5) + (1+2-4+5) + (1+3-4+5) + (2+3-4+5)$$
$$\quad + (1+2-3+4+5)$$
$$= f_4 + f_4 + (5) + (5+5+5+5) - (5+5+5+5+5+5) + (5+5+5+5) + (5)$$
$$= 44 + 44 + 5(1+4-6+4+1)$$
$$= 108.$$

The 5s in the third-last line are grouped by whether they come from computing $\mu(L)$ for a list L of length 1, 2, 3, 4, or 5.

(b) We have $\binom{4}{0} = \binom{4}{4} = 1$, $\binom{4}{1} = \binom{4}{3} = 4$, and $\binom{4}{2} = 6$. Then

$$2f_4 + 5\left(\binom{4}{0} + \binom{4}{1} - \binom{4}{2} + \binom{4}{3} + \binom{4}{4}\right) = 2(44) + 5(1+4-6+4+1) = 108$$

which is the value we obtained for f_5.

2. Our computation of f_5 in Problem 1(a) above gives us some hints as to how to prove this result. Let's pay close attention to this situation first before embarking on the general proof.

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In the solution to 1(a) we arrived at the expression

$$f_5 = 2f_4 + (5) + (5 + 5 + 5 + 5) - (5 + 5 + 5 + 5 + 5 + 5) + (5 + 5 + 5 + 5) + (5).$$

The number of 5s in the first set of parentheses is the number of sublists of $\{1, 2, 3, 4, 5\}$ of length 1 that contain 5. There is only one such list, namely $\{5\}$.

The number of 5s in the second set of parentheses is the number of sublists of $\{1, 2, 3, 4, 5\}$ of length 2 that contain 5. Every such list is of the form $\{k, 5\}$, where $1 \leq a \leq 4$. There are $4 = \binom{4}{1}$ ways to choose an element a from $\{1, 2, 3, 4\}$, so there are four sublists of length 2 that contain 5.

There are six 5s in the third set of parentheses. This is the number of sublists of $\{1, 2, 3, 4, 5\}$ of length 3 that contain 5. Every such sublist is of the form $\{a, b, 5\}$ where $a < b$ and a and b belong to the set $\{1, 2, 3, 4\}$. The number of ways we can choose two distinct elements a and b from $\{1, 2, 3, 4\}$ is precisely the binomial coefficient $\binom{4}{2} = 6$.

Similarly, there are four 5s in the fourth set of parentheses because there are $\binom{4}{3} = 4$ ways to select three elements a, b, c from $\{1, 2, 3, 4\}$ to create a sublist $\{a, b, c, 5\}$ of length 4.

Finally there is $\binom{4}{4} = 1$ way to choose four elements from $\{1, 2, 3, 4\}$ to create a list of length 5 containing 5, which is the list $\{1, 2, 3, 4, 5\}$.

Notice that the third set of parentheses is negated, and all the others are left as they are. Let's look at this a little more closely. Here is what the 3-sign sum of lists containing a 5 look like, for lists of length 1, 2, 3, 4, and 5 respectively:

$$\mu(\{5\}) = 5$$
$$\mu(\{a, 5\}) = a + 5$$
$$\mu(\{a, b, 5\}) = a + b - 5$$
$$\mu(\{a, b, c, 5\}) = a + b - c + 5$$
$$\mu(\{a, b, c, d, 5\}) = a + b - c + d + 5.$$

The only time the 5 appears negated is when computing the 3-sign sum of a list of length 3.

Great! Let's look at the general case now. We wish to relate f_{n+1} to f_n. We can split up the computation of f_{n+1} into summing up the 3-sign sums of sublists of $\{1, \ldots, n+1\}$ that contain $n + 1$, and those that do not contain $n + 1$.

The sum of all the $\mu(L)$ where $n + 1$ is not in L is simply summing up $\mu(L)$ for L a sublist of $\{1, \ldots, n\}$, which is precisely f_n. Therefore we can write

$$f_{n+1} = f_n + S$$

where S is the sum of all 3-sign sums of sublists that contain $n + 1$. Let's look at an arbitrary term from the sum S. For what follows, we must allow the empty list $\emptyset$ to be a sublist of $\{1, \ldots, n\}$ to be a sublist, and we define $\mu(\emptyset) = 0$.

A sublist of length k that contains $n + 1$ is of the form $\{a_1, a_2, \ldots, a_{k-1}, n + 1\}$ where $\{a_1, a_2, \ldots, a_{k-1}\}$ is a sublist of $\{1, \ldots, n\}$. So

$$\mu(\{a_1, a_2, \ldots, a_{k-1}, n+1\}) = \begin{cases} \mu(\{a_1, a_2, \ldots, a_{k-1}\}) - (n + 1) & \text{if } k \text{ is a multiple of 3,} \\ \mu(\{a_1, a_2, \ldots, a_{k-1}\}) + (n + 1) & \text{if } k \text{ is not a multiple of 3.} \end{cases}$$

Observe that $\mu(\{a_1, a_2, \ldots, a_{k-1}\})$ is the 3-sign sum of a sublist of $\{1, \ldots, n\}$.

Every sublist of $\{1, \ldots, n+1\}$ that contains $n+1$ gives rise to a sublist of $\{1, \ldots, n\}$ by removing $n+1$ from the list. Furthermore, every sublist of $\{1, \ldots, n\}$ gives rise to a sublist of $\{1, \ldots, n+1\}$ containing $n+1$ by simply appending $n+1$ on to the end of the list. Therefore, when computing the sum of all 3-sign sums of sublists of $\{1, \ldots, n+1\}$ containing $n+1$, we are left with the sum of all 3-sign sums of sublists of $\{1, \ldots, n\}$ plus some number of $(n+1)$s. More formally we have

$$S = f_n + M(n+1)$$

for some integer M. To figure out the value of M, we must pay attention to the length of each sublist. Sublists of $\{1, \ldots, n+1\}$ containing $n+1$ with length a multiple of three contribute -1 to the value of M, and sublists of $\{1, \ldots, n+1\}$ containing $n+1$ whose length is not a multiple of three contribute $+1$. So, in order to figure out what M is, we must count the number of sublists containing $n+1$ of each length.

The number of sublists of $\{1, \ldots, n+1\}$ of length k containing $n+1$ is equal to the number of sublists of $\{1, \ldots, n\}$ of length $k-1$. The number of sublists of $\{1, \ldots, n\}$ of length $k-1$ is the binomial coefficient $\binom{n}{k-1}$. To write down an expression for M, we must first introduce the integer b_t. Let t be an integer and define

$$b_t = \begin{cases} 1 & \text{if } t+1 \text{ is not a multiple of 3} \\ -1 & \text{if } t+1 \text{ is a multiple of 3.} \end{cases}$$

Let C_k be the number of sublists of $\{1, \ldots, n+1\}$ of length k that contain $n+1$. Then

$$M = b_0 C_1 + b_1 C_2 + \cdots + b_n C_{n+1}.$$

But from our discussion above we have $C_k = \binom{n}{k-1}$. Therefore,

$$M = b_0 \binom{n}{0} + b_1 \binom{n}{1} + \cdots + b_n \binom{n}{n}.$$

Putting all of this together we have

$$f_{n+1} = f_n + f_n + M(n+1)$$
$$= 2f_n + (n+1)\left(b_0 \binom{n}{0} + b_1 \binom{n}{1} + \cdots + b_n \binom{n}{n} \right)$$

completing the proof.

3. (a) We have

$$\begin{aligned} \alpha^3 &= (\alpha^2)\alpha \\ &= (-1-\alpha)\alpha \\ &= -\alpha - \alpha^2 \\ &= 1. \end{aligned}$$

(b) Using the fact that $\alpha^3 = 1$ we have
$$\alpha^8 + \alpha^4 + \alpha^9 = (\alpha^3)(\alpha^3)(\alpha^2) + (\alpha^3)(\alpha) + (\alpha^3)^3$$
$$= \alpha^2 + \alpha + 1$$
$$= -1 + 1$$
$$= 0.$$

(c)
$$(1 + \alpha) - (1 + \alpha)^2 = (1 + \alpha) - (-\alpha^2)^2$$
$$= (1 + \alpha) - (\alpha^4)$$
$$= 1 + \alpha - \alpha \qquad \text{since } \alpha^3 = 1$$
$$= 1.$$

4. (a) Using the binomial theorem we have
$$g_n(1) = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$
$$g_n(\alpha) = \binom{n}{0} + \binom{n}{1}\alpha + \binom{n}{2}\alpha^2 + \cdots + \binom{n}{n}\alpha^n$$
$$g_n(\alpha^2) = \binom{n}{0} + \binom{n}{1}\alpha^2 + \binom{n}{2}\alpha^4 + \cdots + \binom{n}{n}\alpha^{2n}.$$

Since $\alpha^3 = 1$ we can rewrite the second and third expressions as
$$g_n(\alpha) = \binom{n}{0} + \binom{n}{1}\alpha + \binom{n}{2}\alpha^2 + \binom{n}{3} + \binom{n}{4}\alpha + \binom{n}{5}\alpha^2 + \binom{n}{6} + \cdots$$
$$g_n(\alpha^2) = \binom{n}{0} + \binom{n}{1}\alpha^2 + \binom{n}{2}\alpha + \binom{n}{3} + \binom{n}{4}\alpha^2 + \binom{n}{5}\alpha + \binom{n}{6} + \cdots.$$

It is tempting to just take the sum of these two expressions. This would give every third term a coefficient of 2, and every other term a coefficient of $\alpha^2 + \alpha = -1$. A promising development! After all, we want a sum of binomial coefficients where every third term is different from the other two. However, if we simply sum $g_n(\alpha)$ and $g_n(\alpha^2)$, the (approximately) one-third of terms that are different from the rest are in the wrong spot! To remedy this, let's shift where the α and α^2 appear by multiplying each equation by an appropriate power of α. We have
$$\alpha g_n(\alpha) = \binom{n}{0}\alpha + \binom{n}{1}\alpha^2 + \binom{n}{2} + \binom{n}{3}\alpha + \binom{n}{4}\alpha^2 + \binom{n}{5} + \binom{n}{6}\alpha + \cdots$$
$$\alpha^2 g_n(\alpha^2) = \binom{n}{0}\alpha^2 + \binom{n}{1}\alpha + \binom{n}{2} + \binom{n}{3}\alpha^2 + \binom{n}{4}\alpha + \binom{n}{5} + \binom{n}{6}\alpha^2 + \cdots.$$

Much better! Taking the sum we get
$$\alpha g_n(\alpha) + \alpha^2 g_n(\alpha^2) = \binom{n}{0}(\alpha + \alpha^2) + \binom{n}{1}(\alpha^2 + \alpha) + 2\binom{n}{2}$$
$$+ \binom{n}{3}(\alpha + \alpha^2) + \binom{n}{4}(\alpha^2 + \alpha) + 2\binom{n}{5} + \cdots$$
$$= -\binom{n}{0} - \binom{n}{1} + 2\binom{n}{2} - \binom{n}{3} - \binom{n}{4} + 2\binom{n}{5} + \cdots.$$

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Adding $g_n(1)$ to this gives

$$g_n(1) + \alpha g_n(\alpha) + \alpha^2 g_n(\alpha^2) = 3\binom{n}{2} + 3\binom{n}{5} + 3\binom{n}{8} + \cdots .$$

We can now start with $g_n(1)$ and subtract an appropriate multiple of $g_n(1) + \alpha g_n(\alpha) + \alpha^2 g_n(\alpha^2)$ to get what we want.

$$g_n(1) - \frac{2}{3}\left(g_n(1) + \alpha g_n(\alpha) + \alpha^2 g_n(\alpha^2)\right)$$

$$= \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$

$$- 2\left(\binom{n}{2} + \binom{n}{5} + \binom{n}{8} + \cdots\right)$$

$$= \binom{n}{0} + \binom{n}{1} - \binom{n}{2} + \binom{n}{3} + \binom{n}{4} - \binom{n}{5} + \binom{n}{6} + \cdots$$

$$= b_0\binom{n}{0} + b_1\binom{n}{1} + \cdots + b_n\binom{n}{n}.$$

We can now finally conclude that

$$A = \frac{1}{3}$$
$$B = -\frac{2}{3}\alpha$$
$$C = -\frac{2}{3}\alpha^2.$$

(b) We have

$$\begin{aligned}
d_n &= \frac{1}{3}g_n(1) - \frac{2}{3}\alpha g_n(\alpha) - \frac{2}{3}\alpha^2 g_n(\alpha^2) \\
&= \frac{1}{3}(2^n) - \frac{2}{3}\alpha(1+\alpha)^n - \frac{2}{3}\alpha^2(1+\alpha^2)^n \qquad \text{since } g_n(x) = (1+x)^n \\
&= \frac{1}{3}(2^n) - \frac{2}{3}\alpha(-\alpha^2)^n - \frac{2}{3}\alpha^2(-\alpha)^n \\
&= \frac{1}{3}\left(2^n + 2\alpha^{2n+1}(-1)^{n+1} + 2\alpha^{2+n}(-1)^{n+1}\right) \\
&= \frac{2}{3}\left(2^{n-1} + (-1)^{n+1}(\alpha^{2n+1} + \alpha^{2+n})\right).
\end{aligned}$$

Let $r_n = (-1)^{n+1}(\alpha^{2n+1} + \alpha^{2+n})$. The behaviour of $(-1)^{n+1}$ depends on the remainder of n when divided by 2. The behaviour of powers of α depends on the remainder of the power when divided by 3. With this in mind, let's analyse the behaviour of r_n for different remainders of n when divided by $2 \times 3 = 6$.

When $n = 6k$ for some integer k we have

$$\begin{aligned}
r_{6k} &= (-1)^{6k+1}(\alpha^{12k+1} + \alpha^{6k+2}) \\
&= -(\alpha + \alpha^2) \\
&= 1.
\end{aligned}$$

When $n = 6k + 1$ we have

$$r_{6k+1} = (-1)^{6k+2}(\alpha^{12k+3} + \alpha^{6k+3})$$
$$= 2.$$

For $n = 6k + 2$,

$$r_{6k+2} = (-1)^{6k+3}(\alpha^{12k+5} + \alpha^{6k+4})$$
$$= -(\alpha^2 + \alpha)$$
$$= 1.$$

The remaining cases are given by

$$r_{6k+3} = (-1)^{6k+4}(\alpha^{12k+7} + \alpha^{6k+5}) = \alpha + \alpha^2 = -1$$
$$r_{6k+4} = (-1)^{6k+5}(\alpha^{12k+9} + \alpha^{6k+6}) = -(1+1) = -2$$
$$r_{6k+5} = (-1)^{6k+6}(\alpha^{12k+11} + \alpha^{6k+7}) = \alpha^2 + \alpha = -1.$$

So, $d_n = \frac{2}{3}(2^{n-1} + r_n)$, where r_n is given by the following table:

n	r_n
$6k$	1
$6k + 1$	2
$6k + 2$	1
$6k + 3$	-1
$6k + 4$	-2
$6k + 5$	-1

5. Putting all the parts together we have the recursive formula

$$f_{n+1} = 2f_n + \frac{2(n+1)}{3}(2^{n-1} + r_n)$$

where r_n is given by the table above. From 1(a) we know $f_5 = 108$. Applying the recursive formula repeatedly gives

$$f_6 = 2f_5 + \frac{2(6)}{3}(2^4 + r_5) = 2(108) + \frac{2(6)}{3}(2^4 + (-1)) = 276.$$
$$f_7 = 2f_6 + \frac{2(7)}{3}(2^5 + r_6) = 2(276) + \frac{2(7)}{3}(2^5 + 1) = 706.$$
$$f_8 = 2f_7 + \frac{2(8)}{3}(2^6 + r_7) = 2(706) + \frac{2(8)}{3}(2^6 + 2) = 1\,764.$$
$$f_9 = 2f_8 + \frac{2(9)}{3}(2^7 + r_8) = 2(1\,764) + \frac{2(9)}{3}(2^7 + 1) = 4\,302.$$
$$f_{10} = 2f_9 + \frac{2(10)}{3}(2^8 + r_9) = 2(4\,302) + \frac{2(10)}{3}(2^8 + (-1)) = 10\,304.$$
$$f_{11} = 2f_{10} + \frac{2(11)}{3}(2^9 + r_{10}) = 2(10\,304) + \frac{2(11)}{3}(2^9 + (-2)) = 24\,348.$$
$$f_{12} = 2f_{11} + \frac{2(12)}{3}(2^{10} + r_{11}) = 2(24\,348) + \frac{2(12)}{3}(2^{10} + (-1)) = 56\,880.$$

If you were to compute f_{12} directly from the definition, you would have had to compute the 3-sign sum of $4\,096$ lists. What we have done in this Problem of the Month is much less work, especially if you wanted to compute something larger like f_{30}. If you were to compute f_{30} directly from the definition, you would need to compute over 1 billion 3-sign sums! That's a lot of sums.