Problem of the Month

Problem 3: Three-sign sums

December 2024

Hint

1. Ignoring the empty sublist, there are fifteen sublists of $\{1, 2, 3, 4\}$. However, seven of those are sublists of $\{1, 2, 3\}$. So, when computing f_4, you have already done almost half the work when you computed f_3! For the remaining half of the work, notice that $\mu(\{a, b, c\}) = \mu(\{a, b\}) - c$ and $\mu(\{a, b, c, d\}) = \mu(\{a, b, c\}) + d$.

   See if you can then use your work for f_4 to reduce the work required for you to compute f_5. Alternatively, ignoring the empty sublist, there are thirty-one sublists of $\{1, 2, 3, 4, 5\}$, you can just roll up your sleeves and compute the 3-sign sums of all of them!

2. Split up your computation of f_{n+1} into adding up the 3-sign sums of those sublists of $\{1, \ldots, n+1\}$ that contain $n+1$, and those that do not contain $n+1$. When adding up the 3-sign sums of those sublists that contain $n+1$, notice that $\mu(\{a, b, n+1\}) = \mu(\{a, b\}) - (n+1)$ and $\mu(\{a, b, c, n+1\}) = \mu(\{a, b, c\}) + (n+1)$.

3. We have $\alpha^2 + \alpha = -1$. Rearranging we get

   $$-\alpha = 1 + \alpha^2, \quad -\alpha^2 = 1 + \alpha, \quad \text{and} \quad \alpha^2 + \alpha + 1 = 0.$$

   Also, remember that $\alpha^4 = \alpha(\alpha^3)$.

4. (a) Use the binomial theorem to expand $g_n(1)$, $g_n(\alpha)$ and $g_n(\alpha^2)$, keeping in mind your answer from 3(a). What happens if you multiply $g_n(\alpha)$ and $g_n(\alpha^2)$ by powers of α?

   Compute $g_n(1) + g_n(\alpha) + g_n(\alpha^2)$. This isn't quite what you want, but can you adjust the terms somehow to get what you want?

   (b) You should find that d_n involves powers of (-1) in terms of n, and powers of α in terms of n. Powers of (-1) depend on what the remainder of the power is when divided by 2. How do powers of α behave? You may have to split your computation of d_n up into 6 cases.

5. Combine your answers from 1(a), 2, and 4(b).