



Problem of the Month

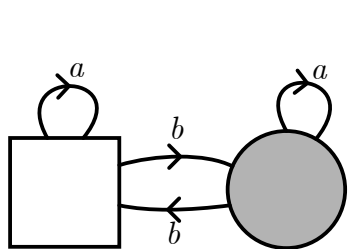
Problem 2: Manuals for Robots

Nov 2024

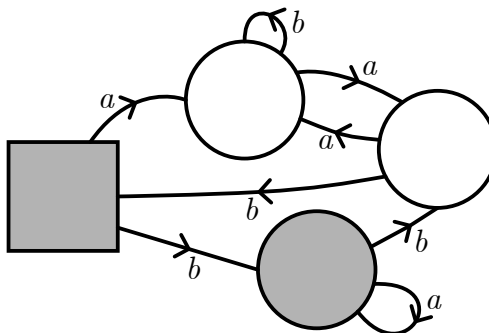
A *course* for a robot is a finite set of places (denoted by circles or squares) with arrows labelled a and b between places, satisfying the following conditions:

1. For every place s , there is exactly one arrow labelled a and one arrow labelled b which starts at s .
2. There is exactly one place called the *starting* place, denoted by a square.
3. Some positive number of places are called *finishing* places, and they are denoted by being shaded. The starting place can be a finishing place.
4. The course is *connected*. That is, there is a way to get from the starting place to any other place by following (possibly several) arrows.

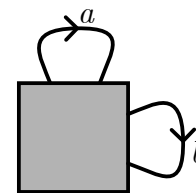
While there is exactly one arrow labelled a and one arrow labelled b starting at every place, there may be multiple arrows labelled a ending at a particular place, or none at all! Take a moment to check that the following three examples are indeed courses.



Course 1



Course 2



Course 3

For a particular course, we can give a robot instructions as to how to move from place to place in the form of a string. A *string* is a finite sequence of a 's and b 's (for example, $abaa$).

For each string, the robot begins at the starting place, and follows the arrows in the order they appear in the string (reading from left to right), moving from place to place.

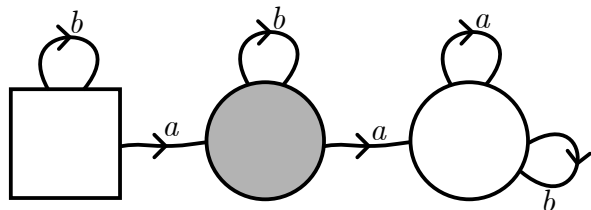
If the robot ends at a finishing place, then the string is *accepted*. Otherwise, the string is *rejected*. Notice that by Condition 1 in the definition of a course, the string completely determines which places the robot visits.

The set of accepted strings is the *manual* for that course. For example, the manual for Course 3 above is the set of all strings. Convince yourself that the manual for Course 1 is the set of strings with an odd number of b 's.

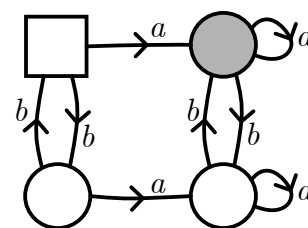


Problems

- Decide which of the following strings belong to the manual for Course 2 from the first page: abb , aab , $baaaa$, $abba$.
- Describe the manuals for the following courses:



(a)



(b)

- Construct a course whose manual is the set of three strings $\{a, ab, bbb\}$.
 - Construct a course whose manual is the set of all strings where the number of a 's in the string is a multiple of 3.
- For this problem we need to introduce some language and notation.

The *length* of a string is the number of characters in the string. For example, $aaba$ has length 4. The string of length 0 is called the *empty string* and we denote it by e .

If x and y are strings, then xy is the *concatenation* of the two strings. The string x^n is x concatenated with itself n times. For example, if $x = bab$, $y = ba$, and $z = aab$, then $xy^3z = babbababaaab$. Define x^0 to be the empty string e .

A *substring* of a string x is a string consisting of consecutive letters from x . For example, ab , ba , e , and $baba$ are all substrings of $baba$. The string bba is not a substring of $baba$.

- Suppose a course has k places, and let w be a string in its manual of length at least k . Show that there is a substring y of w with length at least 1 and at most k so that when it is deleted from w , the remaining word is also in the manual.
- Suppose a course has k places, and let w be a string in its manual of length at least k . Show that you can write $w = xyz$ with the property that $y \neq e$, the length of y is at most k , and for all natural numbers n , $xy^n z$ is also in the manual.
- Prove that the manual for a course cannot be the set of all strings with length equal to a perfect square.
- Prove that the manual for a course cannot be the set of all strings that are palindromes. (A *palindrome* is a string that is the same when written backwards. The strings $abba$ and aaa are palindromes, but ba is not.)