Problem of the Month

Problem 1: A bit of binary

October 2024

When we ordinarily write an integer, we write it in base-10, using digits from the set

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}.$$

For example, when we write 743 what we mean is

$$743 = 7 \times 10^2 + 4 \times 10^1 + 3 \times 10^0.$$

However, there is nothing special about 10, and we can use other numbers as a base! Integers can also be written in base-2, and we call this writing an integer in *binary*. When writing an integer in binary, we use powers of 2 instead of powers of 10, and we use the "digits" from the set $\{0, 1\}$ ("digits" is in scare quotes because binary digits are called *bits*). We are going to explore some problems related to writing numbers in binary.

How do we write a number, say 121, in binary? Well, it's not that different to writing it in base-10. First, find the largest power of 2 which is less than or equal to 121, which is $2^6 = 64$. Subtracting this power gives us $121 - 64 = 57$. Then we repeat with 57, and continue this way until we are left with only a power of 2. With 121 this process looks like this:

$$\begin{aligned} 121 &= 2^6 + 57 \\ 57 &= 2^5 + 25 \\ 25 &= 2^4 + 9 \\ 9 &= 2^3 + 1 \\ 1 &= 2^0 \end{aligned}$$

Once we are done with this process, we conclude that $121 = 2^6 + 2^5 + 2^4 + 2^3 + 2^0$. To be completely explicit,

$$121 = 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0.$$

More compactly, we write this as $121 = [1111001]_2$.

This process works great for natural numbers, but it also works great for all real numbers! Typically, we write real numbers in base-10. For example, $\pi = 3.1415\ldots$ means

$$\pi = 3 \times 10^0 + 1 \times 10^{-1} + 4 \times 10^{-2} + 1 \times 10^{-3} + 5 \times 10^{-4} + \cdots .$$

To write π in binary, we follow the procedure above: find the highest power of 2 less than or equal to π, subtract it, and repeat. The first few steps look like this:

$$\begin{aligned} \pi &= 2^1 + (\pi - 2) \\ \pi - 2 &= 2^0 + (\pi - 3) \\ \pi - 3 &= 2^{-3} + \left(\pi - \frac{25}{8}\right) \end{aligned}$$

Therefore, $\pi = 1 \times 2^1 + 1 \times 2^0 + 0 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + \cdots$ or more succinctly, $\pi = [11.001\ldots]_2$ (take a moment and prove that these calculations are correct, that is, for example, that 2^{-3} is indeed the largest power of 2 less than or equal to $\pi - 3$).

Before we get started on the questions, here are a couple of important facts you can take for granted without proof.

Fact 1: The binary expansions $[0.a_1a_2\cdots a_k0\overline{1}]_2$ and $[0.a_1a_2\cdots a_k1]_2$ represent the same number. This is due to the fact that $\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = 1$. As a result, we never write a binary number ending in an infinite string of 1's. This is analogous to the fact that the decimal expansions $0.\overline{9}$ and 1 are decimal expansions of the same number (that number being the number 1).

Fact 2: With Fact 1 taken care of, every real number has a unique binary expansion.

Problems

1. (a) Compute the binary expansion of 279.

   (b) Let k be a positive integer. Compute the binary expansion of $2^k - 1$.

   (c) Let $\sqrt{3} = [a_0.a_1a_2a_3a_4\ldots]_2$. Compute a_0, a_1, a_2, a_3, and a_4.

2. In base-10, $\frac{1}{7} = 0.\overline{142857}$. In this question we will find the binary expansion of $\frac{1}{7}$, which will also be repeating.

   (a) Find a pair of positive integers k and n so that $2^k \cdot \frac{1}{7} = \frac{1}{7} + n$.

   (b) Let $\frac{1}{7} = [0.a_1a_2a_3\ldots]_2$ (why is the only thing to the left of the decimal point a 0?). Using your values for k and n from part (a), write down binary expansions of $2^k \cdot \frac{1}{7}$ and $\frac{1}{7} + n$ in terms of the a_i.

   (c) Compute the binary expansion of $\frac{1}{7}$. It should look like $[0.\overline{a_1a_2\cdots a_t}]$ for some t.

   (d) Compute the binary expansion of $\frac{3}{11}$.

3. Let p be a prime. Prove that when $\sqrt{p}$ is written in binary, there are infinitely many 1's and infinitely many 0's.

4. The *floor* of a real number x is denoted $\lfloor x \rfloor$, and is the largest integer n so that $n \leq x$. Prove that the sequence

   $$\lfloor\sqrt{2}\rfloor, \lfloor 2\sqrt{2}\rfloor, \lfloor 3\sqrt{2}\rfloor, \lfloor 4\sqrt{2}\rfloor, \lfloor 5\sqrt{2}\rfloor \ldots$$

   contains infinitely many powers of 2.