Grade 7/8 Math Circles

Solving Systems of Equations Using Matrices

Systems of Equations

A **variable** is a placeholder for an unknown numerical value in an equation. Usually, a variable is represented by a letter in the English alphabet, like x, or Greek alphabet, like ϕ.

A **coefficient** is a numerical or constant quantity that multiplies a variable. Normally, we forgo writing the multiplication symbol and place the coefficient right in front of a variable (i.e. $4 \times x = 4x$).

A **system of equations** is two or more equations that share the same variables. For example,

$$x + y = 3$$
$$y = 2x + 3$$

is a system of equations.

Systems of equations are useful because they allow us to solve for the values of multiple variables, that would be impossible with only one equation.

Example 1

What are the values of x and y in the equation $x + y = 3$?

Solution 1

We see that the values $x = 3$ and $y = 0$ give $x + y = 3$, as desired. But, we see that $x = 0$ and $y = 3$ also gives $x + y = 3$. The table below gives a few more values of x and y for the equation $x + y = 3$:

x	0	1	2	3	4	5	...
y	3	2	1	0	-1	-2	...

Thus, we have that there are an infinite number of possible values of x and y in the equation $x + y = 3$.

Here a few changes in notation that will be used in this lesson:

- In many cases of multiplication between two numbers a and b, instead of writing $a \times b$ we often write it as either $a(b)$ or $(a)(b)$, which are all equivalent (i.e. $2 \times 3 = 2(3) = (2)(3)$).
- In the case where we have an expression of the form $a(b + c)$, we can use the **distributive**

property to write it as $ab + ac$ (i.e. $2(5 + 7) = 2(5) + 2(7)$). This also true for any number of values inside the brackets, i.e. $a(b + c + d + e) = a(b) + a(c) + a(d) + a(e)$.

We have the following definitions for the types of systems of equations:

- A system is called **underdetermined** if there are less equations than there are variables. These types of systems have infinitely many solutions.
- A system is called **overdetermined** if there are more equations than there variables. These types of systems can have infinitely many solutions, one solution, or no solutions.
- A system is called **balanced** if there are the same number of equations and variables. These types of systems can have infinitely many solutions, one solution, or no solutions.

Example 1 is the case of an underdetermined system of equations, since there are 2 variables and only 1 equation. In order to determine a **unique** solution to a system of equations, we must make sure that there is at least as many equations as there are variables.

Our initial strategy for solving systems of equations will be **substitution**, which follows these steps:

1. Isolate one of the variables in one equation.
2. Take the expression for the variable from step 1, and substitute it into the other equation(s).
3. Repeat steps 1 and 2 until one variable remains and determine the value of that variable.
4. Use the result from step 3 to solve for the rest of the variables.

Example 2

Determine the values of x and y in the system of equations:

$$x + y = 3$$
$$y = 2x + 3$$

Solution 2

We see that our first step is already done in the second equation with $y = 2x + 3$. We then substitute this value of y into the first equation, which gives:

$$
\begin{aligned}
x + y &= 3 \\
x + 2x + 3 &= 3 && \text{(substitute } y = 2x + 3 \text{ into equation)} \\
3x + 3 &= 3 && \text{(collect } x \text{ terms)} \\
3x + 3 - 3 &= 3 - 3 && \text{(subtract 3 from both sides)} \\
3x &= 0 \\
\frac{3x}{3} &= \frac{0}{3} && \text{(divide both sides by 3)} \\
x &= 0
\end{aligned}
$$

We can now substitute the value $x = 0$ into the second equation to get the value of y:

$$
\begin{aligned}
y &= 2x + 3 \\
y &= 2(0) + 3 \\
y &= 0 + 3 \\
y &= 3
\end{aligned}
$$

Thus, we have $x = 0$ and $y = 3$ is the unique solution to the system of equations. We know that this is the unique system to the problem because no other values of x and y satisfy both equations.

Activity 1

Use the steps above to determine a unique solution to the system of equations:

$$
\begin{aligned}
2x + y &= 4 \\
x - y &= 5
\end{aligned}
$$

Our strategy above will only truly work if the system of equations has a unique solution. If the system is underdetermined, step 3 may or may not produce a value for a variable, but step 4 will never produce values for the all the rest of the variables. If there are a no solutions, there will be many contradictions of the values of variables throughout the steps. The strategy can also work for systems of any number of variables and equations.

Activity 2

Observe the two systems of equations below:

(1)	(2)
$x + y = 1$	$x + y + z = 8$
$x - y = 2$	$2x - z = 18 - y$
$x + 2y = 3$	$3z + 2y - x = 1$

(a) Determine if each system is underdetermined, overdetermined, or balanced. Explain.

(b) Is there a unique solution to (1)? If so, state the solution. If not, how many solutions are there? Show your work.

(c) Is there a unique solution to (2)? If so, state the solution. If not, how many solutions are there? Show your work.

Additionally, sometimes systems of equations are not explicitly stated, and instead have to be created based on information given in the question. This is very common in questions that use systems of equations in real-life situations. The following example will show you how to work through these kinds of problems.

Example 3

Suppose a jar contains marbles of the colours: blue, green, yellow, and red. We are given the following information about the number of marbles:

- The sum of the number of blue marbles and the number of red marbles is 38.
- The sum of the number of green marbles and the number of red marbles is 44.
- The sum of the number of yellow marbles and the number of red marbles is 52.
- There are 100 marbles in total.

How many marbles are there of each colour?

Solution 3

We start by defining a variable to represent the number of marbles for each colour. We have b for blue, g for green, y for yellow, and r for red. We then use the information above to create equations to represent the situation. This gives us the following four equations:

$$(1)\ b + r = 38 \qquad (2)\ g + r = 44 \qquad (3)\ y + r = 52 \qquad (4)\ b + g + y + r = 100$$

We see that the first three equations can all be rewritten in the following ways:

$$(1)\ b = 38 - r \qquad (2)\ g = 44 - r \qquad (3)\ y = 52 - r$$

We can then substitute all three of these equations into the last equation to get:

$$\begin{aligned} b + g + y + r &= 100 \\ (38 - r) + (44 - r) + (52 - r) + r &= 100 \\ 134 - 2r &= 100 \\ -2r &= -34 \\ r &= 17 \end{aligned}$$

Finally, we substitute in the value $r = 17$ into the remaining equations to get:

$$(1)\ b = 38 - 17 = 21 \qquad (2)\ g = 44 - 17 = 27 \qquad (3)\ y = 52 - 17 = 35$$

Thus, there are 21 blue marbles, 27 green marbles, 35 yellow marbles, and 17 red marbles in the jar.

Matrices

A **matrix**, or **matrices** (plural), is a rectangular array of numbers, symbols, or expressions, arranged in rows and columns. Matrices have many applications that are very useful in advanced mathematics, but we will focus on their applications for systems of linear equations. Below is an example of a matrix with 2 rows and 3 columns:

$$\begin{bmatrix} 3 & 2 & 8 \\ 1 & 7 & 4 \end{bmatrix}$$

Elementary Row Operations

The **elementary row operations**, or **EROs**, are the three operations that can be performed on any one matrix.

- **Row Swap**: switching the positions of two rows in the matrix (i.e. $R_1 \leftrightarrow R_2$)
- **Scalar Multiplication**: multiplying the elements in a row by a non-zero constant (i.e. $3 \times R_1$)
- **Row Sum**: Adding the multiple of the elements of one row to the elements of another row (i.e. $R_1 - R_2$)

Performing any of these operations on a matrix does not change the matrix; it remains equivalent to the original matrix.

Example 4

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_1 \leftrightarrow R_2} \begin{bmatrix} 4 & 7 \\ 2 & 3 \end{bmatrix} \quad \text{(Row 1 and Row 2 swap places)}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{matrix} \xrightarrow{2 \times R_1} \\ \rightarrow \end{matrix} \begin{bmatrix} 4 & 6 \\ 4 & 7 \end{bmatrix} \quad \text{(multiply the elements in Row 1 by 2)}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{matrix} \rightarrow \\ \xrightarrow{R_2 - R_1} \end{matrix} \begin{bmatrix} 2 & 3 \\ 2 & 4 \end{bmatrix} \quad \text{(subtract the elements in R2 by the elements in R1)}$$

Reduced Row Echelon Form

The **reduced row echelon form**, or **RREF**, of a matrix is the simplest form of a matrix, similar to a fraction reduced to lowest terms. A matrix is only considered to be in RREF if all of the following are true:

1. The first non-zero element in each row is 1 (called a leading 1).
2. Each leading 1 is in a column to the right of the leading 1 in the previous row.
3. Any rows with all 0 elements, are at the bottom of the matrix.
4. If a column has a leading 1, then all other elements in that column are 0.

Some examples of matrices in RREF are:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

and some examples of matrices not in RREF are:

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

Activity 3

For each of the four matrices above that are not in RREF, determine which of the four conditions of a matrix in RREF are not true.

Example 5

What is the RREF of the matrix $\begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix}$?

Solution 5

Our strategy is to first establish a leading 1 in Row 1

$$\begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix} \xrightarrow[R_2 \leftrightarrow R_1]{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 1 \\ 2 & 7 \end{bmatrix} \xrightarrow[R_2 - 2\times R_1]{} \begin{bmatrix} 1 & 1 \\ 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}\times R_2]{} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus, we have that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the RREF of $\begin{bmatrix} 2 & 7 \\ 1 & 1 \end{bmatrix}$.

Activity 4

Determine the RREF of $\begin{bmatrix} 2 & 6 & 4 \\ 1 & 2 & 5 \\ 2 & 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & -1 \\ 4 & 1 & 3 \\ 5 & 3 & 2 \end{bmatrix}$ using EROs. Show your work.

Solving Systems of Equations

Another method of solving systems of equations is by using matrices. This can be done by rewriting the system of equations as a special kind of matrix called an **augmented matrix**. For example, the system of equations:

$$\begin{aligned} 2x - y &= 3 \\ -4x + 7y &= 5 \end{aligned}$$

can be written as

$$\left[\begin{array}{cc|c} 2 & -1 & 3 \\ -4 & 7 & 5 \end{array}\right]$$

Here, the line between the second and third columns represents the '=' in the equations. Each column on the left side of the line represents a variable, and the column on the right side of the line represents a constant value. Each row represents an equation. The elements of the matrix are coefficients of each variable in each equation. For example, the top left element of the matrix is the coefficient of x in the first equation, which we can see is 2.

In general, for a system of equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m \end{aligned}$$

where x_1, x_2, ..., x_n are the variables, a_{11}, a_{12}, ..., a_{mn} are the coefficients, c_1, c_2, ..., c_m are the constants, m is the number of equations, and n is the number of variables. This can be rewritten as the augmented matrix:

$$\begin{matrix} x_1 & x_2 & \dots & x_n & \\ \end{matrix}$$
$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & c_1 \\ a_{21} & a_{22} & \dots & a_{2n} & c_2 \\ & & \vdots & & \\ a_{m1} & a_{m2} & \dots & a_{mn} & c_m \end{array}\right]$$

The variables are written above to indicate which column corresponds to each.

If the system of equations is underdetermined, the left side of the augmented matrix will have more columns than rows. If the system of equations is overdetermined, the left side of the augmented matrix will have more rows than columns. If the system of equations is balanced, the left side of the matrix will have the same number of rows and columns.

Once we have our system of equations written as an augmented matrix, all we need to do to solve the system is to transform the matrix into its RREF using the same steps as before. The only difference is that the last column does not need a leading 1.

Once the augmented matrix is in RREF, we can determine the number of solutions.

- If there are any rows with all 0 elements (except for the last column), then there are no solutions.
- If any of the rows with leading 1s has a non-zero element (except for the last column), then there are infinitely many solutions.
- If all the elements in a row with a leading 1 are 0 (except for the last column), and every row without a leading 1 has all 0 elements (including last column), then there is a unique solution, and the value of each variable is the element in the last column that is in the same row as its corresponding leading 1.

Example 6

Solve the problem from Example 3 using matrices.

Solution 6

Once again, we define variables to represent the number of marbles for each colour: b for blue, g for green, y for yellow, and r for red; which gives the following equations:

(1) $b + r = 38$ (2) $g + r = 44$ (3) $y + r = 52$ (4) $b + g + y + r = 100$

Now, we use the general form above to create the following augmented matrix:

$$\begin{array}{cccc} b & g & y & r \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 1 & 1 & 1 & 1 & 100 \end{array}\right]$$

Now we transform the augmented matrix to RREF using the same strategy as before.

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 1 & 1 & 1 & 1 & 100 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{R_4 - R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 0 & 1 & 1 & 0 & 62 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{R_4 - R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 0 & 0 & 1 & -1 & 18 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{R_4 - R_3} \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 0 & 0 & 0 & -2 & -34 \end{array}\right] \begin{array}{c} \rightarrow \\ \rightarrow \\ \rightarrow \\ \xrightarrow{-2 \times R_4} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 38 \\ 0 & 1 & 0 & 1 & 44 \\ 0 & 0 & 1 & 1 & 52 \\ 0 & 0 & 0 & 1 & 17 \end{array}\right] \begin{array}{c} \xrightarrow{R_1 - R_4} \\ \xrightarrow{R_2 - R_4} \\ \xrightarrow{R_3 - R_4} \\ \rightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 21 \\ 0 & 1 & 0 & 0 & 27 \\ 0 & 0 & 1 & 0 & 35 \\ 0 & 0 & 0 & 1 & 17 \end{array}\right]$$

Thus, we have $b = 21$, $g = 27$, $y = 35$ and $r = 17$. So, there are 21 blue marbles, 27 green marbles, 35 yellow marbles, and 17 red marbles in the jar.