Grade 6 Math Circles
Probability

Introduction

A **probability** is the likelihood of something happening. It is commonly represented by a percentage, like 25% or 50%, which can also be written as a decimal, like 0.25 or 0.5. For simplicity, we often write the probability of events as a decimal (or fraction) between 0 and 1, where 0 represents certainty the event will not occur (0%) and 1 represents certainty the event will occur (100%).

An **event** is an outcome or result of an experiment to which we assign a probability. Examples of events are getting heads in a coin flip, rolling a 7 with dice, or drawing an Ace from a deck of cards.

For an event A, we denote the probability of A by $P(A)$. One way to calculate the probability of an event is to divide the number of outcomes where the event occurs, denoted by $|A|$, by the total number of possible outcomes, denoted by S. This gives the following formulas:

$$P(A) = \frac{|A|}{S} \quad \text{or} \quad |A| = S \times P(A) \quad \text{or} \quad S = \frac{|A|}{P(A)}$$

Example 1

In a standard deck of 52 cards there are 4 possible suits: **Hearts**, **Diamonds**, **Clubs** and **Spades**; and 13 possible values: **Ace**, **2**, **3**, **4**, **5**, **6**, **7**, **8**, **9**, **10**, **Jack**, **Queen** and **King**. For each suit, each of the 13 values occurs exactly once.
Suppose we randomly draw a card from the deck. What is the probability that the card is a **Diamond**?

Solution 1

There are 52 cards total, so $S = 52$, and there are 13 **Diamonds** in the deck, so $|A| = 13$. So, the probability of the card being a **Diamond**, or $P(A)$, is $\frac{|A|}{S} = \frac{13}{52} = 0.25$ or 25%.

Activity 1

Use the three formulas above to answer the following questions.

(a) What is the probability of drawing a 7 from a standard deck of 52 cards?

(b) What is the probability of rolling an odd number on a standard 6-sided die ('die' is singular for 'dice')?

(c) If the event A occurs 20 times and has a probability of 0.05, how many total outcomes are there?

(d) If the event B has a probability of 0.3 and there are a total of 350 outcomes, how many times does B occur?

The **complement** of an event A is denoted by $\bar{A}$ and defined as every outcome that is not in A. (i.e. for a coin flip if A is 'heads' then $\bar{A}$ is 'tails'). This gives us the following formulas:

$$|\overline{A}| = S - |A| \quad \text{or} \quad |A| = S - |\overline{A}| \quad \text{or} \quad |A| + |\overline{A}| = S$$

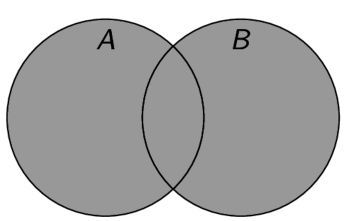
and

$$P(\overline{A}) = 1 - P(A) \quad \text{or} \quad P(A) = 1 - P(\overline{A}) \quad \text{or} \quad P(A) + P(\overline{A}) = 1$$

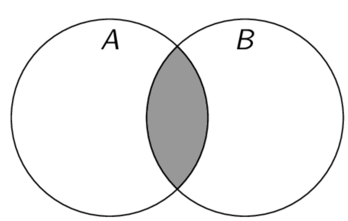
Union and Intersection

For two events A and B, it is often useful to find the probability of at least one of the events occurring. We call this the **union** of A and B, and it is denoted by $A \cup B$ or $B \cup A$. It is also often useful to find the probability of both events occurring. We call this the **intersection** of A and B, and it is denoted by $A \cap B$ or $B \cap A$. If we were to represent events A and B as a Venn Diagram, then $A \cup B$ and $A \cap B$ would be the shaded regions seen below.

$A \cup B$



$A \cap B$



So, $A \cup B$ occurs if either A occurs or B occurs, and $A \cap B$ occurs only when both A and B occur. Because of this, the union of events can be thought of in terms of the **OR** operator, and the intersection of events can be thought of in terms of the **AND** operator. This also leads to the following restrictions for the probabilities of $A \cup B$ and $A \cap B$:

$$P(A \cup B) \geq P(A) \quad \text{and} \quad P(A \cup B) \geq P(B) \quad \text{and} \quad P(A \cup B) \leq P(A) + P(B)$$

$$P(A \cap B) \leq P(A) \quad \text{and} \quad P(A \cap B) \leq P(B)$$

Example 2

Suppose we flip a coin twice, with the event A = first flip is Tails, and the event B = second flip is Heads. What is $A \cup B$ and $A \cap B$?

Solution 2

In this case, $A \cup B$ = the first flip is Tails OR the second flip is Heads, and $A \cap B$ = the first flip is Tails AND the second flip is Heads.

We also have the following formulas that show the connection between the probabilities of the union and intersection of events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{and} \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

which can also be written as:

$$P(A) = P(A \cup B) + P(A \cap B) - P(B) \quad \text{and} \quad P(B) = P(A \cup B) + P(A \cap B) - P(A)$$

Additionally, there is no limit to how many events we can have in a union or an intersection. All of the following unions and intersections are completely valid:

$$A \cup B \cup C \qquad A \cap B \cap C$$
$$A \cup B \cup C \cup D \qquad A \cap B \cap C \cap D$$
$$A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup \ldots \qquad A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap \ldots$$

Note that '. . . ' at the end implies that there are an infinite number of events in the union/intersection.

Example 3

Suppose we draw a card from a standard deck of 52 cards with the following events: A = the card is a **Spade**, and B = the card is a face-card (Jack, Queen, King). Determine $P(A \cup B)$ and $P(A \cap B)$.

Solution 3

Our first step is to determine $P(A)$ and $P(B)$. We already have that $S = 52$. Then, since there are 13 **Spades** in the deck we have $|A| = 13$. Also, since there are 12 face-cards in the deck (3 for each suit) we have $|B| = 12$. So, we have that $P(A) = \frac{|A|}{S} = \frac{13}{52}$ and $P(B) = \frac{|B|}{S} = \frac{12}{52}$. Now we will determine $P(A \cap B)$, which is the probability that the card is a **Spade** and a face-card. Since there are 3 face-cards for each suit, there are only 3 cards in the deck that are both a **Spade** and a face-card, so $|A \cap B| = 3$. Then, we have that $P(A \cap B) = \frac{|A \cap B|}{S} = \frac{3}{52}$. Now we can use one of the formulas above, which gives:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{13}{52} + \frac{12}{52} - \frac{3}{52} = \frac{13 + 12 - 3}{52} = \frac{22}{52} = \frac{11}{26}$$

Thus, $P(A \cup B) = \frac{11}{26}$ and $P(A \cap B) = \frac{3}{52}$.

Similar to any event, unions and intersections also have complements. The formulas for the complement of the union of events and the complement of the intersection of events is given below:

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \text{and} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

These two formulas are more commonly known as **DeMorgan's Laws**.

Activity 2

Use DeMorgan's Laws to solve the following questions.

(a) If the probability of the intersection of A and B is 0.3, what is the probability of the union of the complement of A and the complement of B?

(b) If the probability of the intersection of the complement of A and the complement of B is 0.55, what is the probability of the union of A and B?

Disjoint Events

Events are said to be **disjoint**, or **mutually exclusive**, if it is impossible for them to occur at the same time. For example, when flipping a coin it is impossible to get both Heads and Tails on one coin flip. If we had the events H = Heads and T = Tails, then we would say that H and T are disjoint. This gives us the following results for the probabilities of the union and intersection of disjoint events A and B:

$$P(A \cup B) = P(A) + P(B) \qquad \text{and} \qquad P(A \cap B) = 0$$

Activity 3

Determine whether the following events are disjoint or not. If the events are disjoint, give the probability of $A \cup B$.

(a) A coin is flipped with the events: A = Heads, and B = Tails.

(b) A coin is flipped twice with the events: A = first flip is Heads, and B = second flip is Tails.

(c) A card is drawn from a standard deck of 52 cards with the events: A = card is a 10, and B = card is a face-card.

Independent and Dependent Events

Independent events are events that do not affect one another, meaning that the results do not depend on each other. For example, the outcomes of multiple coin flips are independent events because the result of each coin flip does not depend on any of the other results. This also means that the probability of independent events are not affected by each other, giving us the formulas:

$$P(A \cap B) = P(A) \times P(B) \quad \text{or} \quad P(A) = \frac{P(A \cap B)}{P(B)} \quad \text{or} \quad P(B) = \frac{P(A \cap B)}{P(A)}$$

when A and B are independent events.

Dependent events are events that do affect one another, meaning that the results (and probabilities) are influenced by previous results. For example, if we are drawing from a bag of different coloured marbles and we pick a red marble the first time, and then go to draw a second marble without putting the red marble back in the bag (without replacement), then we have dependent events. This is because there is one fewer red marble in the bag than before, meaning the probability for each marble getting picked has changed.

Activity 4

Determine if the following events are independent or dependent. If they are independent then solve for the probability of the intersection, $A \cap B$.

(a) When flipping a coin, $A =$ first flip is Heads, and $B =$ second flip is Tails.

(b) When flipping a coin, $A =$ first flip is Heads, and $B =$ second flip is Heads.

(c) When rolling a 6-sided die, $A =$ value is even, and $B =$ value is 2.

(d) In a student council election, there are 10 candidates, with 4 of them in grade 7 and 6 of them in grade 8. There are two positions available: President and Treasurer; with $A =$ the President is in grade 7, and $B =$ the Treasurer is in grade 8.

Monty Hall Problem

Example 4

Suppose you are on a game show and the host presents you with 3 identical doors to choose from. They tell you that behind one of this doors is $\$1,000,000$ and there is nothing behind the other two doors. You will receive the contents behind whichever door you choose. Once you make your choice, the host opens one of the two remaining doors that has nothing behind it. Once this is done, the host asks if you want to stay with the door you picked initially, or swap it with the one door remaining. What should you do in order to give yourself the best chances of winning the money?

Solution 4

Statistically, swapping doors will win you the money $\frac{2}{3}$ of the time.

Here is the link to the video that gives an explanation of the Monty Hall Problem:

Monty Hall Problem Explanation