Grade 9/10 Math Circles

Linear Diophantine Equations Part 1 - Problem Set

1. Calculate the following gcds using the Euclidean algorithm.
   (a) $\gcd(12, 57)$
   (b) $\gcd(212, 37)$
   (c) $\gcd(4389, 2919)$

2. For each of the following linear Diophantine equations, find a solution where x and y are integers, or explain why one does not exist.
   (a) $4389x + 2919y = 21$
   (b) $4389x + 2919y = 231$
   (c) $212x - 37y = 1$
   (d) $12x + 57y = 124$
   (e) $12x + 57y = 423$

3. Can 1000 be expressed as the sum of two integers, one of which is divisible by 11 and the other by 17? If so, find examples of such integers. If not, explain why.

4. Can 1000 be expressed as the sum of two integers, one of which is divisible by 9 and the other by 12? If so, find examples of such integers. If not, explain why.

5. Use the Euclidean algorithm to find a solution to $25x + 10y = 215$, the linear Diophantine equation from Example 1 in the Lesson. Does your answer make sense in the context of the problem? If not, how can we find a solution that does make sense?

6. Find an integer n, which, when divided by 78 leaves a remainder of 37, and when divided by 29 leaves a remainder of 17.

7. (a) Suppose a, b, and c are given integers. If $ax^2 + by^2 = c$ has a solution where x and y are integers, is it true that $\gcd(a, b)$ divides c? Why or why not?
   (b) Suppose a, b, and c are given integers. If $\gcd(a, b)$ divides c, is it true that $ax^2 + by^2 = c$ always has a solution where x and y are integers? Why or why not?

8. (a) Use the division algorithm and the important fact to show that $\gcd(k + 1, k) = 1$ for any integer $k \geq 1$.

(b) Use the division algorithm and the important fact to show that

$$\gcd(7k + 6, 6k + 5) = 1$$

for any integer $k \geq 1$.