Grade 9/10 Math Circles
Linear Diophantine Equations Part 2 - Solutions

Exercise Solutions

Exercise 1

Determine the complete solution to the linear Diophantine equation $8x + 9y = 3$.

Exercise 1 Solution

To determine the complete solution, we first need to find one solution.
By inspection, we can see that $x = -3$, $y = 3$ is a solution to this linear Diophantine equation.
Thus, we can use $x_0 = -3$, $y_0 = 3$ to generate the complete solution.
That is, the complete solution is

$$x = -3 + n\left(\frac{b}{d}\right), \quad y = 3 - n\left(\frac{a}{d}\right)$$

Substituting $a = 8$, $b = 9$, and $d = \gcd(8, 9) = 1$:

$$x = -3 + n\left(\frac{9}{1}\right), \quad y = 3 - n\left(\frac{8}{1}\right)$$

This simplifies to $x = -3 + 9n$, $y = 3 - 8n$.
Thus, the complete solution to the linear Diophantine equation $8x + 9y = 3$ is

$x = -3 + 9n,\ \ y = 3 - 8n$, where n is any integer.

Exercise 2

Determine the complete solution to the linear Diophantine equation $4182x + 3689y = 102$.

Exercise 2 Solution

To determine the complete solution, we first need to find one solution. We will use the techniques we learned in the previous lesson to first find one solution.

We begin by calculating $\gcd(4182, 3689)$ using the Euclidean Algorithm.

$$4182 = 3689 \cdot 1 + 493, \text{ so } \gcd(4182, 3689) = \gcd(3689, 493) \quad (1)$$
$$3689 = 493 \cdot 7 + 238, \text{ so } \gcd(3689, 493) = \gcd(493, 238) \quad (2)$$
$$493 = 238 \cdot 2 + 17, \text{ so } \gcd(493, 238) = \gcd(238, 17) \quad (3)$$
$$238 = 17 \cdot 14 + 0, \text{ so } \gcd(238, 17) = 17 \quad (4)$$

Thus, $\gcd(4182, 3689) = 17$.
Since $102 = 6 \times 17$, we know that there is a solution to the given linear Diophantine equation.
To find a solution, we work backwards through the steps above.
From line (3), we have

$$17 = 493 - 2 \cdot 238$$

Substituting for 238 using (2):

$$17 = 493 - 2(3689 - 7 \cdot 493)$$

Simplifying:

$$17 = 15 \cdot 493 - 2 \cdot 3689$$

Substituting for 493 using (1):

$$17 = 15(4182 - 1 \cdot 3689) - 2 \cdot 3689$$

Simplifying:

$$17 = 15(4182) - 17(3689)$$

Therefore, we can see that $4182(15) + 3689(-17) = 17$.
Multiplying both sides of this equation by 6, we obtain:

$$6[4182(15) + 3689(-17)] = 6(17)$$
$$4182(6 \times 15) + 3689(6 \times (-17)) = 102$$
$$4182(90) + 3689(-102) = 102$$

Thus, a solution to the linear Diophantine equation $4182x + 3689y = 102$ is $x = 90$, $y = -102$.

To determine the complete solution, we calculate:

$$x = 90 + n\left(\frac{b}{d}\right), \quad y = -102 - n\left(\frac{a}{d}\right)$$

Substituting $a = 4182$, $b = 3689$, and $d = 17$:

$$x = 90 + n\left(\frac{3689}{17}\right), \quad y = -102 - n\left(\frac{4182}{17}\right)$$

This simplifies to $x = 90 + 217n$, $y = -102 - 246n$.
Thus, the complete solution to the linear Diophantine equation $4182x + 3689y = 102$ is

$$x = 90 + 217n,\ y = -102 - 246n, \text{ where } n \text{ is any integer.}$$

Problem Set Solutions

1. This problem will step you through determining all non-negative solutions to the linear Diophantine equation $12x + 57y = 423$.

   (a) Use the Euclidean Algorithm to calculate $\gcd(12, 57)$.

   (b) Using part (a), determine a solution to $12x + 57y = 3$.

   (c) Using part (b), determine a solution to $12x + 57y = 423$.

   (d) Using part (c), determine *all* solutions to $12x + 57y = 423$.

   (e) Using your answer in part (d), determine all solutions to $12x + 57y = 423$ with $x \geq 0$ and $y \geq 0$. That is, determine all non-negative solutions to the linear Diophantine equation $12x + 57y = 423$.

Solution:

(a) Using the Euclidean algorithm, we find

$$57 = 12 \cdot 4 + 9 \quad (1)$$

$$12 = 9 \cdot 1 + 3 \quad (2)$$

$$9 = 3 \cdot 3 + 0 \quad (3)$$

Thus, $\gcd(12, 57) = 3$.

(b) To find a solution, we will work backwards through the steps in the Euclidean algorithm in our solution to part (a).

$$\begin{aligned} 3 &= \underline{12} - 1 \cdot \underline{9} && \text{from (2)} \\ &= \underline{12} - 1 \cdot (\underline{57} - 4 \cdot \underline{12}) && \text{from (1)} \\ &= 5 \cdot \underline{12} - 1 \cdot \underline{57} \end{aligned}$$

Thus,

$$12(5) + 57(-1) = 3$$

Thus, one solution to the linear Diophantine equation $12x + 57y = 3$ is $x = 5$, $y = -1$.

(c) In part (b), we found that one solution to the linear Diophantine equation $12x+57y = 3$ is $x = 5$, $y = -1$.

Multiplying both sides of the equation by 141 gives,

$$\begin{aligned} 141(12(5) + 57(-1)) &= 141(3) \\ 12(141 \cdot 5) + 57(141 \cdot (-1)) &= 423 \\ 12(705) + 57(-141) &= 423 \end{aligned}$$

Thus, one solution to the linear Diophantine equation $12x + 57y = 423$ is $x = 705$, $y = -141$.

(d) In part (c), we found that one solution to the linear Diophantine equation $12x+57y = 423$ is $x = 705$, $y = -141$.

To determine the complete solution, we calculate:

$$x = 705 + n\left(\frac{b}{d}\right), \quad y = -141 - n\left(\frac{a}{d}\right)$$

Substituting $a = 12$, $b = 57$, and $d = 3$:

$$x = 705 + n\left(\frac{57}{3}\right), \quad y = -141 - n\left(\frac{12}{3}\right)$$

This simplifies to $x = 705 + 19n$, $y = -141 - 4n$.

Thus, the complete solution to the linear Diophantine equation $12x + 57y = 423$ is $x = 705 + 19n,\ \ y = -141 - 4n$, where n is any integer.

(e) In part (d), we found that the complete solution to the linear Diophantine equation $12x + 57y = 423$ is $x = 705 + 19n,\ \ y = -141 - 4n$, where n is any integer.

Now, we also require $x \geq 0$ and $y \geq 0$. Which values of n will satisfy these conditions?

$$\begin{aligned} x \geq 0 &\implies 705 + 19n \geq 0 \\ &\implies 19n \geq -705 \\ &\implies n \geq -37.105... \end{aligned}$$

So n is an integer and $n \geq -37.105$, which tells us that $n \geq -37$.

Similarly,

$$\begin{aligned} y \geq 0 &\implies -141 - 4n \geq 0 \\ &\implies -4n \geq 141 \\ &\implies n \leq -35.25 \end{aligned}$$

So n is an integer and $n \leq -35.25$, which tells us that $n \leq -36$.

Thus, in order for $x \geq 0$ and $y \geq 0$, we must have $n \geq -37$ and also have $n \leq -36$. Thus, there are 2 possible values for n: -36, -37.

Using $x = 705 + 19n$ and $y = -141 - 4n$, we find there are two solutions to this problem. They are:

- $x = 21$, $y = 3$ (when $n = -36$)
- $x = 2$, $y = 7$ (when $n = -37$)

2. Explain why there is no solution to the linear Diophantine equation from Exercise 2,

$$4182x + 3689y = 102$$

with $x \geq 0$ and $y \geq 0$.

Solution: In Exercise 2, we found that the complete solution to the linear Diophantine equation $4182x + 3689y = 102$ is $x = 90 + 217n$, $y = -102 - 246n$, where n is any integer. Now, we also require $x \geq 0$ and $y \geq 0$. Which values of n will satisfy these conditions?

$$\begin{aligned} x \geq 0 &\implies 90 + 217n \geq 0 \\ &\implies 217n \geq -90 \\ &\implies n \geq -0.4147... \end{aligned}$$

So n is an integer and $n \geq -0.4147$, which tells us that $n \geq 0$.
Similarly,

$$\begin{aligned} y \geq 0 &\implies -102 - 246n \geq 0 \\ &\implies -246n \geq 102 \\ &\implies n \leq -0.4146... \end{aligned}$$

So n is an integer and $n \leq -0.4146$, which tells us that $n \leq -1$.
Thus, in order for $x \geq 0$ and $y \geq 0$, we must have $n \geq 0$ and also have $n \leq -1$. There are no values of n that satisfy both of these inequalities simultaneously. Therefore, there is no solution to the linear Diophantine equation $4182x + 3689y = 102$ with $x \geq 0$ and $y \geq 0$.

3. Determine all possible ways that 1000 can be expressed as the sum of two **positive** integers, one which is divisible by 11 and the other by 17.

Solution:
In Problem Set #3 for Part 1 we were asked to find one way to express 1000 as the sum of two integers, one which is divisible by 11 and the other by 17.
In the solution, we saw that this problem is equivalent to asking if there is a solution to the linear Diophantine equation $11x + 17y = 1000$, and we saw that one solution is $x = -3000$ and $y = 2000$. (So the two numbers are $-33\,000$, which is divisible by 11, and $34\,000$, which is divisible by 17.)
We are now looking to find a solutions where both x and y are positive.

First, we can determine the complete solution by calculating:

$$x = -3000 + n\left(\frac{b}{d}\right), \quad y = 2000 - n\left(\frac{a}{d}\right)$$

Substituting $a = 11$, $b = 17$, and $d = 1$:

$$x = -3000 + 17n, \quad y = 2000 - 11n$$

Thus, the complete solution to the linear Diophantine equation $11x + 17y = 1000$ is $x = -3000 + 17n$, $y = 2000 - 11n$, where n is any integer.

Now, we also require $x > 0$ and $y > 0$. Which values of n will satisfy these conditions?

$$\begin{aligned} x > 0 &\implies -3000 + 17n > 0 \\ &\implies 17n > 3000 \\ &\implies n > 176.47... \end{aligned}$$

So n is an integer and $n > 176.47$, which tells us that $n \geq 177$.

Similarly,

$$\begin{aligned} y > 0 &\implies 2000 - 11n > 0 \\ &\implies -11n > -2000 \\ &\implies n < 181.818... \end{aligned}$$

So n is an integer and $n < 181.818$, which tells us that $n \leq 181$.

Thus, when both $n \geq 177$ and $n \leq 181$, we will have $x \geq 0$ and $y \geq 0$.

Thus, there are 5 possible values for n: 177, 178, 179, 180, 181.

Using $x = -3000 + 17n$ and $y = 2000 - 11n$, we find the five solutions to this problem where the two numbers are non-negative. They are:

- $n = 177$: $x = 9$, $y = 53$, so the two numbers that sum to 1000 are 99 and 901.
- $n = 178$: $x = 26$, $y = 42$, so the two numbers that sum to 1000 are 286 and 714.
- $n = 179$: $x = 43$, $y = 31$, so the two numbers that sum to 1000 are 473 and 527.
- $n = 180$: $x = 60$, $y = 20$, so the two numbers that sum to 1000 are 660 and 340.
- $n = 181$: $x = 77$, $y = 9$, so the two numbers that sum to 1000 are 847 and 153.

4. At a museum, an adult ticket costs \$34 and a student ticket costs \$28. A group visiting the museum spends exactly \$844 on tickets. Determine all possible combinations for the number of adult and student tickets they could have purchased.

Solution: Let a be the number of adult tickets and s be the number of student tickets purchased.
Since an adult ticket costs \$34 and a student ticket costs \$28 and a total of \$844 was spent on tickets, we are interested in solving the linear Diophantine equation

$$34a + 28s = 844$$

We first use the Euclidean Algorithm to calculate $\gcd(34, 28)$:

$$\begin{aligned} 34 &= 28 \cdot 1 + 6 && (1) \\ 28 &= 6 \cdot 4 + 4 && (2) \\ 6 &= 4 \cdot 1 + 2 && (3) \\ 4 &= 2 \cdot 2 + 0 && (4) \end{aligned}$$

So $\gcd(34, 28) = 2$. Since 844 is divisible by 2, there is a solution to this linear Diophantine equation.
We find a solution to $34a + 28s = 2$ by working backwards through our steps from the Euclidean Algorithm:

$$\begin{aligned} 2 &= 6 - 1 \cdot \underline{4} && \text{from (3)} \\ &= 6 - 1(28 - 4 \cdot 6) && \text{from (2)} \\ &= 5 \cdot \underline{6} - 1 \cdot 28 \\ &= 5(34 - 1 \cdot 28) - 1 \cdot 28 && \text{from (1)} \\ &= 5(34) - 6(28) \end{aligned}$$

Therefore, $34(5) + 28(-6) = 2$.
Multiplying both sides by 422:

$$\begin{aligned} 422[34(5) + 28(-6)] &= 422(2) \\ 34(422(5)) + 28(422(-6)) &= 844 \\ 34(2110) + 28(-2532) &= 844 \end{aligned}$$

So a solution is $a = 2110$ and $s = -2532$. Thus, the complete solution is:
$a = 2110 + n(\frac{28}{2}) = 2110 + 14n$ and $s = -2532 - n(\frac{34}{2}) = -2532 - 17n$
Since a and s represent the number of tickets sold, we need $a \geq 0$ and $s \geq 0$.
$a \geq 0$ means that $2110 + 14n \geq 0$, thus $14n \geq -2110$ and $n \geq -150.71...$
Since n is an integer, we need $n \geq -150$.
$s \geq 0$ means that $-2532 - 17n \geq 0$, thus $-17n \geq 2532$ and $n \leq -148.94...$
Since n is an integer, we need $n \leq -149$.
Thus, given the context of the problem, it must be the case that $-150 \leq n \leq -149$.
When $n = -150$, we obtain $a = 2110+14n = 2110+14(-150) = 10$ and $s = -2532-17n = -2532 - 17(-150) = 18$.
When $n = -149$, we obtain $a = 2110+14n = 2110+14(-149) = 24$ and $s = -2532-17n = -2532 - 17(-149) = 1$.
Therefore, there are two possibilities for the tickets purchased. There could have been 10 adult tickets and 18 student tickets purchased, or 24 adult tickets and 1 student ticket purchased.

5. Find the smallest positive integer x so that $157x$ leaves remainder 10 when divided by 24.

Solution:
We need to first solve $157x = 24y + 10$ for integers x and y.
In other words, we need to solve the linear Diophantine equation $157x - 24y = 10$.
We first use the Euclidean Algorithm, to calculate $\gcd(157, 24)$:

$$157 = 24 \cdot 6 + 13 \quad (1)$$
$$24 = 13 \cdot 1 + 11 \quad (2)$$
$$13 = 11 \cdot 1 + 2 \quad (3)$$
$$11 = 2 \cdot 5 + 1 \quad (4)$$
$$2 = 1 \cdot 2 + 0 \quad (5)$$

Working backwards through these steps, we obtain:

$$\begin{aligned} 1 &= \underline{11} - 5 \cdot \underline{2} && \text{from (4)} \\ &= \underline{11} - 5 \cdot (\underline{13} - 1 \cdot \underline{11}) && \text{from (3)} \\ &= 6 \cdot \underline{11} - 5 \cdot \underline{13} \\ &= 6 \cdot (\underline{24} - 1 \cdot \underline{13}) - 5 \cdot \underline{13} && \text{from (2)} \\ &= 6 \cdot \underline{24} - 11 \cdot \underline{13} \\ &= 6 \cdot \underline{24} - 11 \cdot (\underline{157} - 6 \cdot \underline{24}) && \text{from (1)} \\ &= 72 \cdot \underline{24} - 11 \cdot \underline{157} \end{aligned}$$

Therefore, $157(-11) - 24(-72) = 1$.
Multiplying both sides of this equation by 10:

$$\begin{aligned} 10[157(-11) - 24(-72)] &= 10(1) \\ 157(10(-11)) - 24(10(-72)) &= 10 \\ 157(-110) - 24(-720) &= 10 \end{aligned}$$

Therefore, one solution to $157x - 24y = 10$ is $x = -110,\ \ y = -720$.
Thus, the complete solution is
$x = -110 + (\frac{-24}{1})n = -110 - 24n$ and $y = -720 - (\frac{157}{1})n = -720 - 157n$
We are asked for the smallest positive value of x. In order to have $x \geq 0$, we would need $x = -110 - 24n \geq 0$. This implies that $24n \leq -110$, or $n \leq -4.583\ldots$.
Since n is an integer, this implies that we must have $n \leq -5$.
So the smallest possible value for x is when $n = -5$, and we have $x = -110 - 24(-5) = 10$.
Indeed, we can check. When $x = 10$, then $157x = 1570$, which has remainder 10 when divided by 24.

6. Determine the number of ways you can make exactly \$200 using exactly 1000 coins if each coin is a quarter, a dime, or a nickel.

Solution: Let q represent the number of quarters, d represent the number of dimes and n represent the number of nickels.

From the information given in the problem, we have

$$q + d + n = 1000 \quad (1)$$
$$25q + 10d + 5n = 20\,000 \quad (2)$$

Dividing equation (2) by 5, we obtain

$$5q + 2d + n = 4000 \quad (3)$$

Subtracting equation (1) from equation (3), we obtain: $4q + d = 3000$.
Thus, we need to solve the linear Diophantine equation $4q + d = 3000$.
By inspection, one solution to $4q + d = 3000$ is $q = 0$, $d = 3000$.
Therefore, the complete solution is
$q = 0 + (\frac{1}{1})k = k$ and $d = 3000 - (\frac{4}{1})k = 3000 - 4k$, where k is any integer.
Since $q + d + n = 1000$, we have $n = 1000 - q - d = 1000 - k - (3000 - 4k) = 3k - 2000$.
Since q, d and n represent coins, we also have the constraints $q \geq 0$, $d \geq 0$, and $n \geq 0$.
Since $q \geq 0$ and $q = k$, we have $k \geq 0$.
Since $d \geq 0$ and $d = 3000 - 4k$, we have $3000 - 4k \geq 0$. Solving, we find that $4k \leq 3000$, and thus $k \leq 750$.
Since $n \geq 0$ and $n = 3k - 2000$, we have $3k - 2000 \geq 0$. Solving, we find that $3k \geq 2000$, and thus $k \geq 666.\bar{6}$. Since k must be an integer, we need $k \geq 667$.
Thus, $667 \leq k \leq 750$.
Therefore, there are $750 - 667 + 1 = 84$ ways to make exactly \$200 using exactly 1000 coins if each is a quarter, dime, or a nickel.

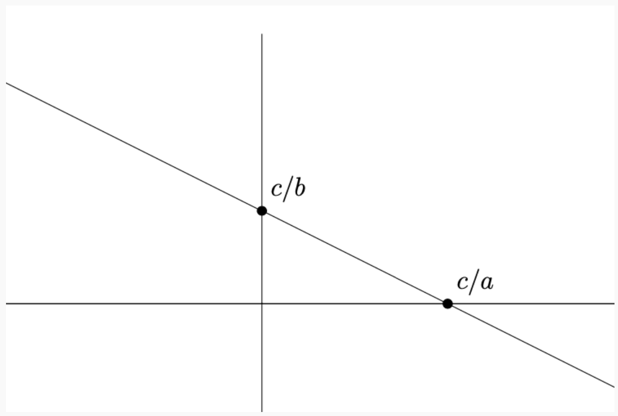
7. Let a, b, and c be positive integers and consider the linear Diophantine equation $ax + by = c$. Show that the number of non-negative integer solutions to this equation cannot exceed $\frac{c}{a}$ or $\frac{c}{b}$.

Solution: Solutions to the linear Diophantine equation $ax + by = c$ correspond to points on the line $y = -\frac{a}{b}x + \frac{c}{b}$.
This line has slope $-\frac{a}{b}$ and y-intercept $\frac{c}{b}$.

We can find the x-intercept of this line by setting $y = 0$ and solving for x:

$$0 = -\frac{a}{b}x + \frac{c}{b} \implies \frac{a}{b}x = \frac{c}{b} \implies x = \frac{cb}{ab} = \frac{c}{a}$$

Therefore, this line has x-intercept $\frac{c}{a}$.
The graph of this line looks something like this:



Notice that the non-negative solutions to the linear Diophantine equation $ax + by = c$ are exactly the integer points that lie on the line segment from $\left(0, \frac{c}{b}\right)$ to $\left(\frac{c}{a}, 0\right)$. How many of these points can there be?
Since $0 \leq x \leq \frac{c}{a}$, there are at most $\frac{c}{a}$ such points.
Likewise, since $0 \leq y \leq \frac{c}{b}$, there are at most $\frac{c}{b}$ such points.
Therefore, the number of integer solutions to the linear Diophantine equation $ax + by = c$ with $x \geq 0$ and $y \geq 0$ cannot exceed $\frac{c}{a}$ or $\frac{c}{b}$