Grade 9/10 Math Circles
An Introduction to Group Theory Part 2 - Solutions

Exercise Solutions

Exercise 1

Let $X = \{1, 2, 3, 4\}$. List all of the permutations of X.

Exercise 1 Solution

The complete list of permutations of X is as follows:

$$(1,2,3,4) \quad (1,2,4,3) \quad (1,3,2,4) \quad (1,3,4,2) \quad (1,4,2,3) \quad (1,4,3,2)$$

$$(2,1,3,4) \quad (2,1,4,3) \quad (2,3,1,4) \quad (2,3,4,1) \quad (2,4,1,3) \quad (2,4,3,1)$$

$$(3,1,2,4) \quad (3,1,4,2) \quad (3,2,1,4) \quad (3,2,4,1) \quad (3,4,1,2) \quad (3,4,2,1)$$

$$(4,1,2,3) \quad (4,1,3,2) \quad (4,2,1,3) \quad (4,2,3,1) \quad (4,3,1,2) \quad (4,3,2,1)$$

Exercise 2

Let $X = \{1, 2, 3, 4, 5\}$ and consider the following two permutations of S_5

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 3 & 5 \end{pmatrix}.$$

Compute the compositions $\sigma \circ \beta$ and $\beta \circ \sigma$.

Exercise 2 Solution

We compute that

$$\sigma \circ \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 5 & 3 & 2 & 4 \end{pmatrix}$$

and
$$\beta \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 2 & 1 & 4 & 3 \end{pmatrix}.$$

Exercise 3

Convince yourself that $(S_3, \circ)$ is a group.

Exercise 3 Solution

We need to convince ourselves that the 3 group axioms hold for S_3 with composition. Let's go through each axiom:

Axiom 1: For all $\sigma, \beta, \alpha \in S_3$, we need $(\sigma \circ \beta) \circ \alpha$ to be the same permutation as $\sigma \circ (\beta \circ \alpha)$. Showing this in a quick way is a bit tough given our current mathematical tools. So, let's do a concrete example to see why this axiom holds. Consider the following permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \quad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

We compute that

$$\sigma \circ \beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{and} \quad \beta \circ \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}.$$

So

$$(\sigma \circ \beta) \circ \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

and

$$\sigma \circ (\beta \circ \alpha) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

We see that $(\sigma \circ \beta) \circ \alpha = \sigma \circ (\beta \circ \alpha)$, as desired.

Axiom 2: The identity element in S_3 is the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}.$$

Given how composition works, if you compose this element with any other element $\sigma \in S_3$, we just get σ back. So indeed, this is the identity element id_{S_3}. For example, we see that

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Axiom 3: We need to show that every element in S_3 has an inverse. Let's show how to construct the inverse of any element in S_3 by using a concrete example. Consider

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

The inverse σ^{-1} of σ is obtained by switching the rows of σ and then reordering the columns so that the first row is $1, 2, 3$. That is,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{\text{switch rows}} \begin{pmatrix} 3 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix} \xrightarrow{\text{reorder columns}} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \sigma^{-1}.$$

And indeed, we see that $\sigma \circ \sigma^{-1} = \text{id}_{S_3} = \sigma^{-1} \circ \sigma$ as desired. We can do this with any element in S_3 to get it's inverse!

Exercise 4

Show that the symmetry group of the square (i.e., the regular polygon with 4 sides) is **not** the same as the symmetric group on $\{1, 2, 3, 4\}$. To get started, consider the following labelling of the square:
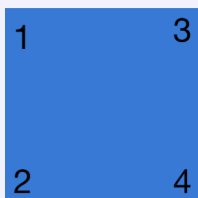
Exercise 4 Solution

In lesson 2 we saw that every symmetry of the equilateral triangle can be identified with a permutation of $\{1, 2, 3\}$. In the exact same way, we can identity every symmetry of the square with a permutation of $\{1, 2, 3, 4\}$. For example, consider the symmetry counter-clockwise rotation by 90 degrees:
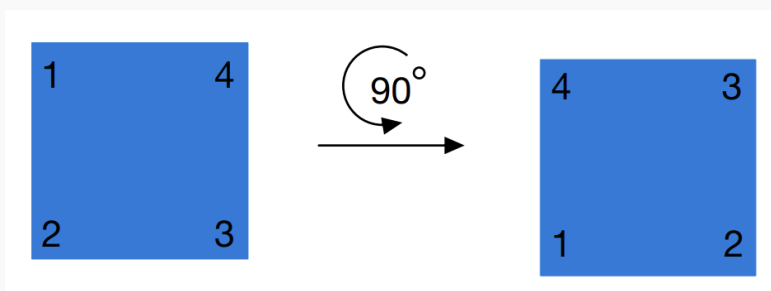
We see that the 1 moves to where 2 was, 2 moves to where 3 was, 3 moves to where 4 was, and 4 moves to where 1 was. Given this, we identify this symmetry with the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

In a similar way, we can identify every symmetry of the square with an element of S_3. However, not every permutation of $\{1, 2, 3, 4\}$ can be identified with a symmetry of the square. One way to see this is to notice that there are more permutations of $\{1, 2, 3, 4\}$ than symmetries of the

square. As seen in the solution for Exercise 1, there are 24 permutations of $\{1, 2, 3, 4\}$. However, there are only 8 symmetries of the square (see Problem 7 Solution in the solutions for Lesson 1). Since the number of symmetries of the square differs from the number of elements in S_3, the symmetry group of the square can not be the same as the symmetric group on $\{1, 2, 3, 4\}$.

Exercise 5

Consider the original situation where only Amy and Professor Farnsworth have switched minds (see Figure 3).

1. Can we get Amy's mind back into Amy's body and Farnsworth's mind back into Farnsworth's body?

2. Further, can (1) be done in such a way that everyone who switches minds ends up with their own mind in the end?

Exercise 5 Solution

The answer to both questions is yes! Let's see why. Recall that our current situation is as follows:
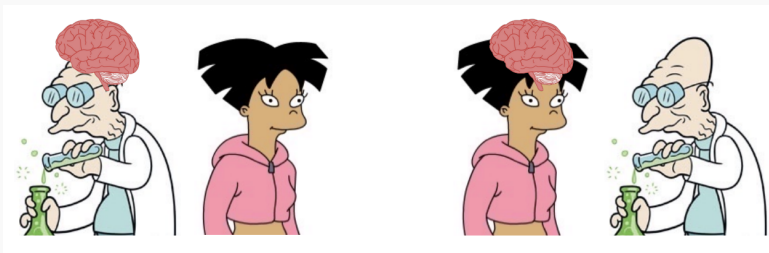
The key idea is to introduce two beings who have never switched minds yet. In the episode they introduce the two characters Sweet and Bubblegum:

Figure 1: Sweet (left) and Bubblegum (right)

We fist get Farnsworth's body and Bubblegum's body to switch minds:



Then we get Amy's body and Sweet's body to switch minds:

Note that Amy's body has switched minds with Farnsworth's body an Sweet's body. So the only switch Amy's body can do is with Bubblegum's body. Similarly, the only switch Farnsworth's body can do is with Sweet's body. These switches result in:
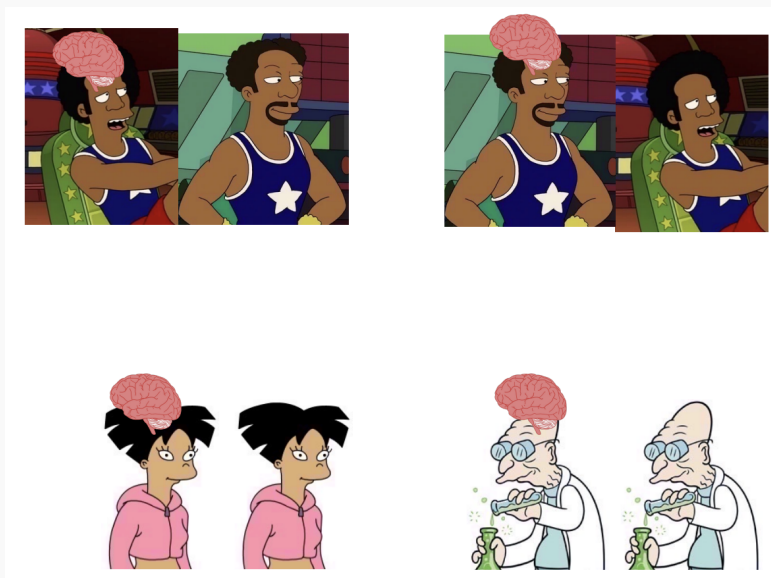
We see that Amy's mind is back in Amy's body, and the same is true for Farnsworth. Since Sweet and Bubblegum's bodies have not switched minds yet, we can do this switch to get everyone's

mind back into their own body:
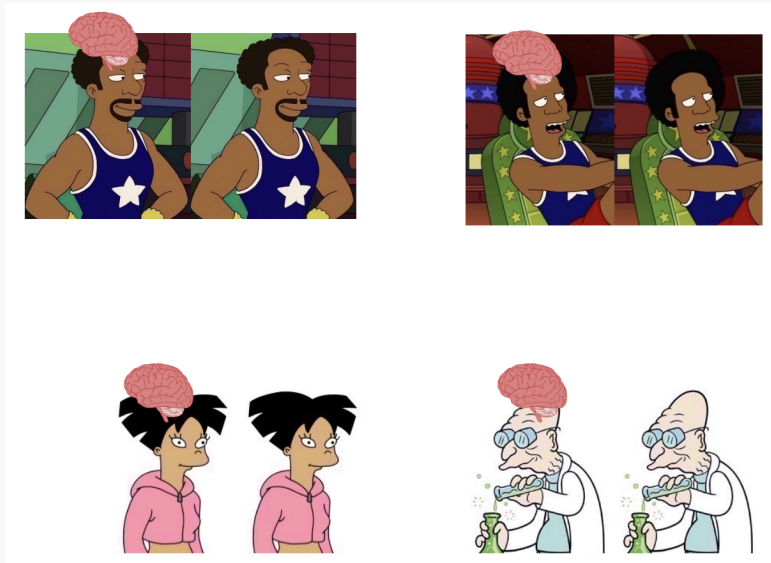
We have successfully answered (1) and (2)! Note that this approach can be generalized to any number of mind switches.

All character photos in this solution set are from: Futurama Fandom. Futurama wiki characters [Online]. Available from: https://futurama.fandom.com/wiki/Characters [Accessed March 24 2023].

Problem Set Solutions

1. Compute the following compositions:

   (a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 4 \end{pmatrix}$

   > *Solution*: We compute that:
   >
   > (a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$
   >
   > (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$
   >
   > (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 2 & 4 & 1 \end{pmatrix}$

2. Compute the inverses of the following permutations:

   (a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

   (c) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix}$

   > *Solution*: As illustrated in the solution to Problem 1, we see that the inverse of (a) is itself. The inverse of (b) is also itself. And the inverse of (c) is
   >
   > $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 6 & 2 & 5 \end{pmatrix}.$$

3. Let P_n be a regular polygon with n sides where $n \geq 4$. Convince yourself that the symmetry group of P_n is not the same as the symmetric group on $\{1, \ldots, n\}$.

   hint: how many elements are in $Sym(P_n)$ and S_n?

   Solution: The argument for Exercise 4 also holds here. In other words, one reason to see why the symmetry group of P_n is not the same as the symmetric group on $\{1, \ldots, n\}$ is to realize that the number of elements in $\mathrm{Sym}(P_n)$ differs from the number of elements in S_n. In fact, there are $2n$ elements in $\mathrm{Sym}(P_n)$ and $n!$ elements in S_n. And we see that $2n < n!$ when $n \geq 4$.