Grade 9/10 Math Circles
An Introduction to Group Theory Part 1 - Problem Set

1. In Example 4 we saw that $(\mathbb{Z}, +)$ is a group. Now consider subtraction $-$ on $\mathbb{Z}$. Convince yourself that subtraction is a binary operation on $\mathbb{Z}$. Show that subtraction on $\mathbb{Z}$ is not associative and use this to conclude that $(\mathbb{Z}, -)$ is not a group.

2. Consider the set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, ...\}$, which is the set containing all non-negative whole numbers. Addition is a binary operation on $\mathbb{N}$. Show that $(\mathbb{N}, +)$ is not a group.

3. Consider the set of even integers

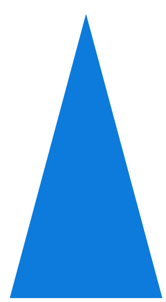
$$2\mathbb{Z} = \{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}.$$

   Convince yourself that addition is a binary operation on $2\mathbb{Z}$. Show that $(2\mathbb{Z}, +)$ is a group.

4. Suppose we are given a group $(G, \bullet)$. Axiom 2 of the group axioms says that $(G, \bullet)$ has an identity element, which we denote by $\text{id}_G \in G$. Show that the identity element id_G is unique. In other words, show that there is exactly one element of G that satisfies the property: for every $a \in G$, $a \bullet \text{id}_G = a = \text{id}_G \bullet a$.

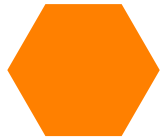
   Hint: If $e \in G$ satisfies the above property, try to show that $e = id_G$.

5. Consider the following triangle:



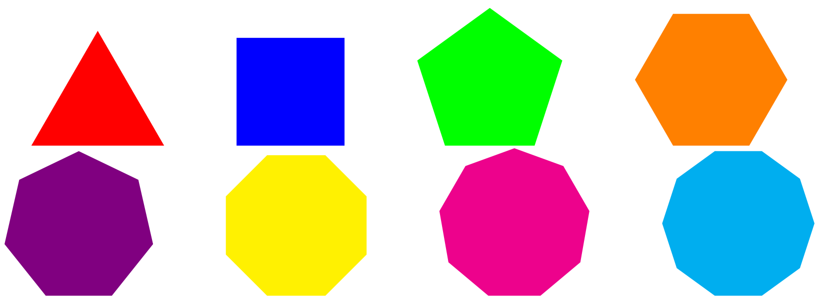
   Write down the symmetries of this triangle and compare them with the symmetries of the equilateral triangle from Exercise 4.

6. Consider the following hexagon:



Write down the symmetries of this hexagon. Then compare the symmetries of this hexagon with the symmetries of the benzene molecule, which were found in Exercise 6. Are they the same in some sense? Or are they different?

7. Consider the following shapes:



These shapes are examples of regular polygons. A polygon is called regular if all of its sides have the same length and all of its angels are the same. For example, an equilateral triangle is a regular polygon. Write down the symmetry group of an arbitrary regular polygon.
hint: Let P_n be a regular polygon with n sides. The goal is to write down the elements of Sym(P_n). In Exercise 4 you computed Sym(P_3), and in Problem 6 you computed Sym(P_6). If you compute Sym(P_n) for small n, say $n \in \{3, 4, 5, 6\}$, do you see a pattern?