Functions, Equations and Polynomials

Toolkit

Functions

A *function* is a set of ordered pairs (x, y), in which for every x value, there is exactly one y value. If we graph the function, then it must pass the *Vertical Line Test*, which says that if we draw a vertical line anywhere on the graph, it will pass through at most one point. We use the notation $f(x) = y$ to state that y is the value of the function that corresponds to x.

Composition of functions is the process of combining two or more functions, where one function is performed first and the result is substituted in place of x into the next function and so on. The composition of functions f and g is written as $f(g(x))$. In the notation $f(g(x))$, the function g is applied first, followed by the function f.

The *inverse* of a function undoes the action of the function. That is, if the function is a set of ordered pairs (x, y), then the inverse will be set of ordered pairs (y, x). If the inverse of a function $f(x)$ is also a function, it is denoted $f^{-1}(x)$.

- $f^{-1}(f(x)) = x$ for all x values in the domain of $f(x)$
- $f(f^{-1}(x)) = x$ for all x values in the domain of $f^{-1}(x)$.

Solving Equations, Inequalities and Systems of Equations

You should be able to solve an equation, an inequality or a system of equations algebraically. When solving a system of equations, the method of elimination or the method of substitution are two methods that can be useful.

Parabolas

The quadratic polynomial $f(x) = ax^2 + bx + c$ (with a,b,c real and $a \neq 0$) has two roots given by the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. These roots are

- real and distinct, if $b^2 - 4ac > 0$
- real and equal, if $b^2 - 4ac = 0$
- distinct and non-real, if $b^2 - 4ac < 0$

The sum of these two roots is $-\dfrac{b}{a}$ and their product is $\dfrac{c}{a}$.

Since $y = ax^2 + bx + c = a\left(x + \dfrac{b}{2a}\right)^2 + \dfrac{4ac - b^2}{4a}$, the parabola's vertex is located at $\left(-\dfrac{b}{2a}, \dfrac{4ac - b^2}{4a}\right)$.

Polynomials

- The *Remainder Theorem* states that when a polynomial $p(x) = a_0 + a_1x^1 + \cdots + a_nx^n$, of degree n, is divided by $(x - k)$ the remainder is $p(k)$.
- The *Factor Theorem*, which follows from the Remainder Theorem, states that $p(k) = 0$ if and only if $(x - k)$ is a factor of $p(x)$.
- A polynomial equation $p(x) = 0$, where $p(x)$ has degree n, has at most n real roots.
- The *Rational Roots Theorem* states that if we have a polynomial $p(x)$ with integer coefficients, then the rational roots of the polynomial are of the form $\frac{q}{r}$, where q is a factor of the constant term of $p(x)$ and r is a factor of the leading coefficient of $p(x)$ (the coefficient of the term of the highest degree).

Sample Problems

1. For each positive real number x, define $f(x)$ to be the number of prime numbers p that satisfy $x \le p \le x + 10$. What is the value of $f(f(20))$?

   Solution

   Let $a = f(20)$. Then $f(f(20)) = f(a)$. To calculate $f(f(20))$, we determine the value of a and then the value of $f(a)$. By definition, $a = f(20)$ is the number of prime numbers p that satisfy $20 \le p \le 30$. The prime numbers between 20 and 30, inclusive, are 23 and 29, so $a = f(20) = 2$. Thus, $f(f(20)) = f(a) = f(2)$. By definition, $f(2)$ is the number of prime numbers p that satisfy $2 \le p \le 12$. The prime numbers between 2 and 12, inclusive, are 2, 3, 5, 7, 11, of which there are 5. Therefore, $f(f(20)) = 5$.

2. If $x^2 - x - 2 = 0$, determine all possible values of $1 - \frac{1}{x} - \frac{6}{x^2}$.

   Solution

   We have
   $$1 - \frac{1}{x} - \frac{6}{x^2} = \frac{x^2 - x - 6}{x^2} = \frac{x^2 - x - 2 - 4}{x^2}$$
   Since $x^2 - x - 2 = 0$, we have
   $$\frac{x^2 - x - 2 - 4}{x^2} = -\frac{4}{x^2}$$
   Factoring, we obtain $x^2 - x - 2 = (x - 2)(x + 1) = 0$. Thus, $x = 2$ or $x = -1$. Therefore, the possible values of the expression are -1 and -4.

3. Given that $x^2 = 8x + y$ and $y^2 = x + 8y$ with $x \ne y$, determine the value of $x^2 + y^2$.

   Solution

   Adding the two equations we obtain $x^2 + y^2 = 9x + 9y$. Subtracting the second equation from the first we obtain $x^2 - y^2 = 7x - 7y$. Factoring both sides we obtain $(x - y)(x + y) = 7(x - y)$. Since $x \ne y$, we have that $x - y \ne 0$ and so we can divide both sides by $x - y$ to obtain that $x + y = 7$. Therefore, $x^2 + y^2 = 9(x + y) = 9(7) = 63$.

4. If the graph of the parabola $y = x^2$ is translated to a position such that its x intercepts are $-d$ and e and its y intercept is $-f$, (where d,e,$f > 0$), show that $de = f$.

   Solution 1 (easy)

   Since the x intercepts are $-d$ and e, the parabola must be of the form $y = a(x + d)(x - e)$. Also, since we have only translated $y = x^2$, it follows that $a = 1$. When $x = 0$, we obtain the y-intercept. Therefore, setting $x = 0$ gives $-f = -de$ and the result follows.

   Solution 2 (harder)

   Let the parabola be $y = ax^2 + bx + c$. Now, as in the first solution, $a = 1$. Then solving for the x- and y-intercepts we find $e = \frac{-b + \sqrt{b^2 - 4c}}{2}$, $-d = \frac{-b - \sqrt{b^2 - 4c}}{2}$ and $-f = c$. Now multiplication of these two expressions gives $-de = \frac{-b - \sqrt{b^2 - 4c}}{2} \cdot \frac{-b + \sqrt{b^2 - 4c}}{2} = \frac{b^2 - b^2 + 4c}{4} = c = -f$ as required.

5. Find all values of x such that $x + \dfrac{36}{x} \geq 13$.

Solution

First, we note that $x \neq 0$. If $x > 0$, we can multiply the inequality by this positive quantity and arrive at $x^2 - 13x + 36 \geq 0$ or $(x-4)(x-9) \geq 0$. We have two cases to consider. The first has $x - 4 \geq 0$ and $x - 9 \geq 0$ and the second has $x - 4 \leq 0$ and $x - 9 \leq 0$.

In the first case, the two inequalities combine to give $x \geq 9$. In the second case, the two inequalities combine to give $x \leq 4$. We also have that $x > 0$, and so this gives $0 < x \leq 4$ or $x \geq 9$.

If $x < 0$, the left side of the inequality is negative, which means it is not greater than 13. Therefore, $0 < x \leq 4$ or $x \geq 9$.

6. If a polynomial leaves a remainder of 5 when divided by $x - 3$ and a remainder of -7 when divided by $x + 1$, what is the remainder when the polynomial is divided by $x^2 - 2x - 3$?

Solution

We observe that when we divide a polynomial by a second degree polynomial the remainder will be a linear polynomial or a constant polynomial. Thus, the division statement becomes

$$p(x) = (x^2 - 2x - 3)q(x) + ax + b \qquad (*)$$

where $p(x)$ is the polynomial, $q(x)$ is the quotient polynomial and $ax + b$ is the remainder. Now we observe that the remainder theorem states that $p(3) = 5$ and $p(-1) = -7$. Also we notice that $x^2 - 2x - 3 = (x-3)(x+1)$. Thus, substituting $x = 3$ and -1 into $(*)$ we obtain

$$p(3) = 5 = 3a + b$$
$$p(-1) = -7 = -a + b$$

Solving the resulting system of equations gives $a = 3$ and $b = -4$. Therefore, the remainder is $3x - 4$.

Problem Set

1. If x and y are real numbers, determine all solutions (x, y) to the system of equations

$$x^2 - xy + 8 = 0$$
$$x^2 - 8x + y = 0$$

2. The parabola defined by the equation $y = (x-1)^2 - 4$ intersects the x-axis at points P and Q. If (a,b) is the midpoint of PQ, what is the value of a?

3. (a) The equation $y = x^2 + 2ax + a$ represents a parabola for all real values of a. Prove that there exists a common point through which all of these parabolas pass, and determine the coordinates of this point.

   (b) The vertices of these parabolas lie on a curve. Prove that this curve is itself a parabola whose vertex is the common point found in part (a).

4. Determine all real values of p and r that satisfy the following system of equations.

$$p + pr + pr^2 = 26$$
$$p^2r + p^2r^2 + p^2r^3 = 156$$

5. A quadratic equation $ax^2 + bx + c = 0$ (where a, b, and c are not zero), has real roots. Prove that a, b, c, in that order, cannot be consecutive terms in a geometric sequence.

6. A quadratic equation $ax^2 + bx + c = 0$ (where a, b, and c are integers and $a \neq 0$), has integer roots. If a, b, c, in that order, are consecutive terms in an arithmetic sequence, solve for the roots of the equation.

7. Solve the following equation for x.

$$(x^2 - 3x + 1)^2 - 3(x^2 - 3x + 1) + 1 = x.$$

8. The parabola $y = (x-2)^2 - 16$ has its vertex at point A and its larger x-intercept at point B. Find the equation of the line through A and B.

9. Solve the equation $(x-b)(x-c) = (a-b)(a-c)$ for x.

10. Given that $x = -2$ is a solution of $x^3 - 7x - 6 = 0$, find the other solutions.

11. Find the value of a such that the equation below in x has real roots, the sum of whose squares is a minimum.

$$4x^2 + 4(a-2)x - 8a^2 + 14a + 31 = 0$$

12. If $f(x) = \dfrac{3x-7}{x+1}$ and $g(x)$ is the inverse of $f(x)$, then determine the value of $g(2)$.

13. If $(-2,7)$ is the maximum point for the function $y = -2x^2 - 4ax + k$, determine k.

14. The roots of $x^2 + cx + d = 0$ are a and b and the roots of $x^2 + ax + b = 0$ are c and d. If a, b, c and d are nonzero, calculate $a + b + c + d$.

15. If $y = x^2 - 2x - 3$, then determine the minimum value of $\dfrac{y-4}{(x-4)^2}$.

16. Suppose that the function g satisfies $g(x) = 2x - 4$ for all real numbers x and that g^{-1} is the inverse function of g. Suppose that the function f satisfies $g(f(g^{-1}(x))) = 2x^2 + 16x + 26$ for all real numbers x. What is the value of $f(\pi)$?