Exponents and Logarithms
Solutions

1. We have $\log_x(2 \cdot 4 \cdot 8) = 1$ which implies that $\log_x(64) = 1$. Therefore, $x = 64$.

2. Since $12 = 2^2 \cdot 3$ it follows that

$$\begin{aligned} 12^{2x+1} &= 2^{3x+7} \cdot 3^{3x-4} \\ 2^{2(2x+1)} \cdot 3^{2x+1} &= 2^{3x+7} \cdot 3^{3x-4} \\ 2^{2(2x+1)-3x-7} &= 3^{3x-4-2x-1} \\ 2^{x-5} &= 3^{x-5} \end{aligned}$$

The graphs of $y = 2^{x-5}$ and $y = 3^{x-5}$ intersect only at $x = 5$ and $y = 1$. (Since $2^z = 3^z$ only for $z = 0$.) Therefore, $x = 5$ is the only solution.

3. This expression equals $\log_{10}\left(\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \cdots \cdot \frac{200}{199}\right) = \log_{10}\left(\frac{200}{2}\right) = \log_{10} 100 = 2$.

4. For the first equation we know that $x, y \neq 0$. The second equation states $xy^{-2} = 3^{-3}$ which gives us $x^3y^{-6} = 3^{-9}$ after cubing both sides. Since neither side of this new equation is 0 we can divide the first equation by this new equation to eliminate x.

$$\begin{aligned} \frac{x^3y^5}{x^3y^{-6}} &= \frac{2^{11}3^{13}}{3^{-9}} \\ y^{11} &= 2^{11} \cdot 3^{22} \\ y^{11} &= 2^{11} \cdot (3^2)^{11} \\ y^{11} &= (2 \cdot 9)^{11} \\ y &= 18. \end{aligned}$$

Therefore, $x = \frac{y^2}{27} = 12$.

5. $\log_8(18) = \log_8 2 + \log_8 9 = \frac{1}{3} + \log_8 3^2 = \frac{1}{3} + 2\log_8 3 = \frac{1}{3} + 2k = 2k + \frac{1}{3}$

6. **Solution 1**

We express the logarithms in exponential form to arrive at: $2x = 2^y$ and $x = 4^y$. Thus,

$$\begin{aligned} 2^y &= 2(4^y) \\ 2^y &= 2(2^{2y}) \\ 2^y &= 2^{2y+1} \\ y &= 2y + 1 \\ y &= -1 \end{aligned}$$

Thus, $x = 4^{-1} = \frac{1}{4}$. Therefore, the point of intersection is $\left(\frac{1}{4}, -1\right)$.

Solution 2

Substituting one equation into the other we obtain

$$\begin{aligned}
\log_2 2x &= \log_4 x \\
\frac{\log 2x}{\log 2} &= \frac{\log x}{\log 4} \\
\frac{\log 2x}{\log 2} &= \frac{\log x}{\log 2^2} \\
\frac{\log 2x}{\log 2} &= \frac{\log x}{2\log 2} \\
2\log 2x &= \log x \\
2\log 2 + 2\log x &= \log x \\
\log x &= -2\log 2 \\
\log x &= \log 2^{-2} \\
\log x &= \log \frac{1}{4}
\end{aligned}$$

Therefore, $x = \frac{1}{4}$. Substituting this value back into either of the original equations and we obtain that $y = -1$. Therefore, the point of intersection is $\left(\frac{1}{4}, -1\right)$.

7. We note first that $x = a^y$ for all points on the curve. The midpoint of AB is given by $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$. Since we draw a horizontal line through the midpoint, $y_3 = \frac{y_1+y_2}{2}$, we have that

$$\begin{aligned}
(x_3)^2 &= (a^{y_3})^2 \\
&= (a^{\frac{y_1+y_2}{2}})^2 \\
&= (a^{y_1})(a^{y_2}) \\
&= x_1x_2.
\end{aligned}$$

8. Given that the graph of the function passes through $(2, 1)$, we know that $a \neq 0$. We have $1 = a(2^r)$ and $4 = a(32^r)$. Since neither side of the first equation is 0, we can divide the second equation by the first to obtain $4 = \frac{32^r}{2^r} = \frac{2^r \cdot 16^r}{2^r} = 16^r$. Therefore, $r = \frac{1}{2}$.

9. Factoring both sides of the equation $2^{x+3} + 2^x = 3^{y+2} - 3^y$ gives

$$\begin{aligned}
(2^3+1)2^x &= (3^2-1)3^y \\
9 \cdot 2^x &= 8 \cdot 3^y \\
3^2 \cdot 2^x &= 2^3 \cdot 3^y \\
2^{x-3} &= 3^{y-2}
\end{aligned}$$

Since x and y are integers, and the only integer power of 2 that is also an integer power of 3 is the number $1 = 2^0 = 3^0$, we have $x = 3$ and $y = 2$.

10. If $f(x) = 2^{4x-2}$, then $f(x) \cdot f(1-x) = 2^{4x-2} \cdot 2^{4(1-x)-2} = 2^{4x-2+4-4x-2} = 2^0 = 1$.

11. Observe that the argument of both logarithms must be positive and so $x > 6$. Now

$$\begin{aligned} \log_5(x-2) + \log_5(x-6) &= 2 \\ \log_5((x-2)(x-6)) &= 2 \\ (x-2)(x-6) &= 25 \\ x^2 - 8x - 13 &= 0 \\ x &= 4 \pm \sqrt{29} \end{aligned}$$

However, since $x > 6$, we have that $x = 4 + \sqrt{29}$.

12. If a, b, c is a geometric sequence, then $\frac{b}{a} = \frac{c}{b}$. It follows that $\log_x\left(\frac{b}{a}\right) = \log_x\left(\frac{c}{b}\right)$ which implies $\log_x b - \log_x a = \log_x c - \log_x b$. Therefore, the logarithms form an arithmetic sequence.

If $\log_x a, \log_x b, \log_x c$ form an arithmetic sequence, then

$$\begin{aligned} \log_x b - \log_x a &= \log_x c - \log_x b \\ \log_x\left(\frac{b}{a}\right) &= \log_x\left(\frac{c}{b}\right) \\ \frac{b}{a} &= \frac{c}{b} \quad \text{since the log function takes on each value only once} \end{aligned}$$

Thus, a, b, c form a geometric sequence.

13. Using exponent rules and arithmetic, we manipulate the given equation:

$$\begin{aligned} 3^{x+2} + 2^{x+2} + 2^x &= 2^{x+5} + 3^x \\ 3^x 3^2 + 2^x 2^2 + 2^x &= 2^x 2^5 + 3^x \\ 9(3^x) + 4(2^x) + 2^x &= 32(2^x) + 3^x \\ 8(3^x) &= 27(2^x) \\ \frac{3^x}{2^x} &= \frac{27}{8} \\ \left(\frac{3}{2}\right)^x &= \left(\frac{3}{2}\right)^3 \end{aligned}$$

Since the two expressions are equal and the bases are equal, then the exponents must be equal, and so $x = 3$.

14. Let $a = \log_{10} x$. Then $(\log_{10} x)^{\log_{10}(\log_{10} x)} = 10\,000$ becomes $a^{\log_{10} a} = 10^4$.

Taking the base 10 logarithm of both sides and using the fact that $\log_{10}(a^b) = b \log_{10} a$, we obtain $(\log_{10} a)(\log_{10} a) = 4$ or $(\log_{10} a)^2 = 4$. Therefore, $\log_{10} a = \pm 2$ and so $\log_{10}(\log_{10} x) = \pm 2$.

If $\log_{10}(\log_{10} x) = 2$, then $\log_{10} x = 10^2 = 100$ and so $x = 10^{100}$.

If $\log_{10}(\log_{10} x) = -2$, then $\log_{10} x = 10^{-2} = \frac{1}{100}$ and so $x = 10^{1/100}$. Therefore, $x = 10^{100}$ or $x = 10^{1/100}$.

15. Note that $x \neq 1$ since 1 cannot be the base of a logarithm. This tells us that $\log x \neq 0$.

Using the fact that $\log_a b = \dfrac{\log b}{\log a}$ and then using other logarithm laws, we obtain the following equivalent equations:

$$\begin{aligned}
\log_4 x - \log_x 16 &= \tfrac{7}{6} - \log_x 8 \\
\frac{\log x}{\log 4} - \frac{\log 16}{\log x} &= \frac{7}{6} - \frac{\log 8}{\log x} \qquad \text{(note that } x \neq 1 \text{, so } \log x \neq 0) \\
\frac{\log x}{\log 4} &= \frac{7}{6} + \frac{\log 16 - \log 8}{\log x} \\
\frac{\log x}{\log(2^2)} &= \frac{7}{6} + \frac{\log(\frac{16}{8})}{\log x} \\
\frac{\log x}{2\log 2} &= \frac{7}{6} + \frac{\log 2}{\log x} \\
\frac{1}{2}\left(\frac{\log x}{\log 2}\right) &= \frac{7}{6} + \frac{\log 2}{\log x}
\end{aligned}$$

Letting $t = \dfrac{\log x}{\log 2} = \log_2 x$ and noting that $t \neq 0$ since $x \neq 1$, we obtain the following equations equivalent to the previous ones:

$$\begin{aligned}
\frac{t}{2} &= \frac{7}{6} + \frac{1}{t} \\
3t^2 &= 7t + 6 \qquad \text{(multiplying both sides by } 6t) \\
3t^2 - 7t - 6 &= 0 \\
(3t+2)(t-3) &= 0
\end{aligned}$$

Therefore, the original equation is equivalent to $t = -\frac{2}{3}$ or $t = 3$.

Converting back to the variable x, we obtain $\log_2 x = -\frac{2}{3}$ or $\log_2 x = 3$, which gives $x = 2^{-2/3}$ or $x = 2^3 = 8$.