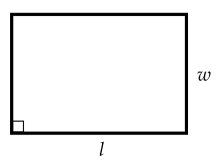
Euclidean Geometry

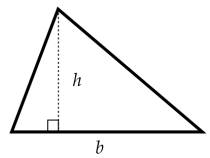
Toolkit

Area

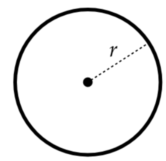
- $A_{\text{Rectangle}} = l \times w$



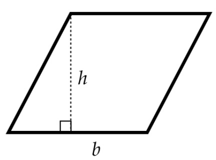
- $A_{\text{Triangle}} = \frac{1}{2}(b \times h)$



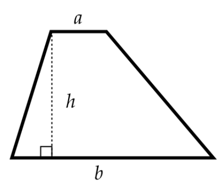
- $A_{\text{Circle}} = \pi r^2$



- $A_{\text{Parallelogram}} = b \times h$



- $A_{\text{Trapezoid}} = \frac{1}{2}(a + b)h$

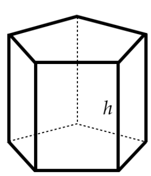


Note: The perimeter of a circle is $2\pi r$.

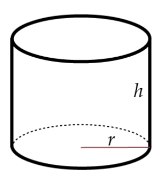
- The area of an equilateral triangle of side length s is $\dfrac{\sqrt{3}s^2}{4}$.
- Heron's formula - The area of a triangle with side lengths $a, b,$ and c is $\sqrt{s(s-a)(s-b)(s-c)}$ where $s = \dfrac{a+b+c}{2}$.

Volume

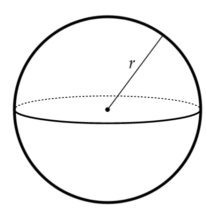
- $V_{\text{Prism}} = A_{\text{Base}} \times h$



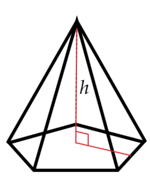
- $V_{\text{Cylinder}} = \pi r^2 h$



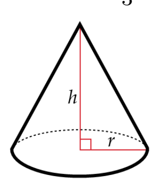
- $V_{\text{Sphere}} = \frac{4}{3}\pi r^3$



- $V_{\text{Pyramid}} = \frac{1}{3}A_{\text{Base}} \times h$



- $V_{\text{Cone}} = \frac{1}{3}\pi r^2 \times h$



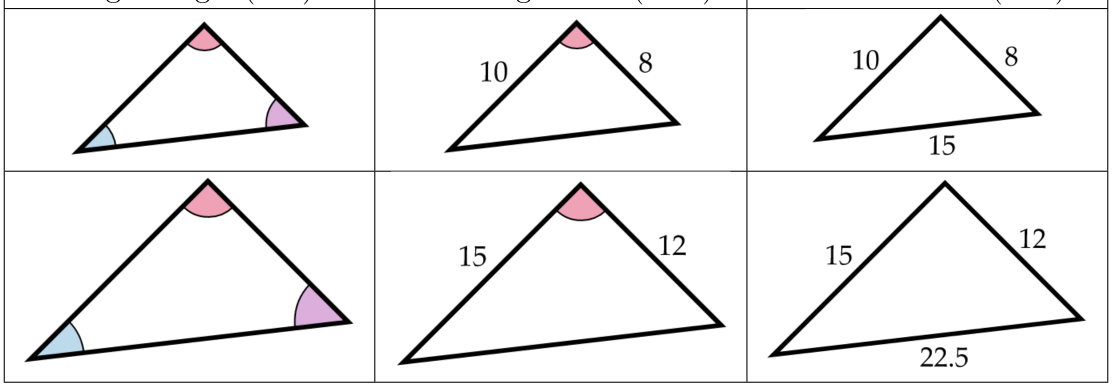
Pythagorean Theorem

- For a right triangle with side lengths, a, b and c, where c is the length of the hypotenuse, we have $a^2 + b^2 = c^2$.
- Ordered triples of integers (a, b, c) which satisfy this relationship are called *Pythagorean Triples*. The triples $(3, 4, 5)$, $(7, 24, 25)$ and $(5, 12, 13)$ are common examples.

Similarity and Congruence

- When two polygons have equal corresponding angles, they are similar. When the corresponding side lengths are also equal, they are congruent.
- Two triangles are similar when they satisfy any of the following rules:
  - Angle-angle similarity (two corresponding angles are equal)
  - Side-angle-side similarity (two pairs of corresponding sides are in the same proportion and the contained angles are equal)
  - Side-side-side similarity (all three pairs of corresponding sides are in the same proportion)

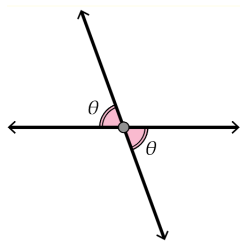
Angle-Angle (AA)	Side-Angle-Side (SAS)	Side-Side-Side (SSS)



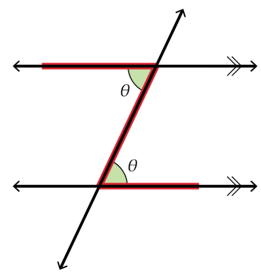
Angles

- For a polygon with n sides, the sum of the interior angles is given by $((n-2)\times 180)^\circ$ or $(180n-360)^\circ$.
- For a regular polygon, all interior angles are equal in measure.
- A trapezoid has two pairs of supplementary angles.
- A parallelogram has opposite angles that are equal and adjacent angles that are supplementary.
- Angles that form a straight line and share a vertex add to 180°.
- Two angles that sum to 90° are called *complementary*.

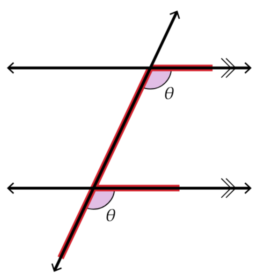
- Opposite angles are equal.



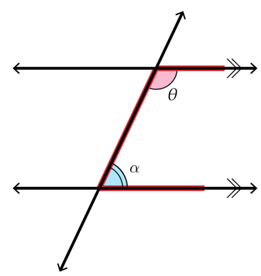
- Alternate angles are equal.



- Corresponding angles are equal.

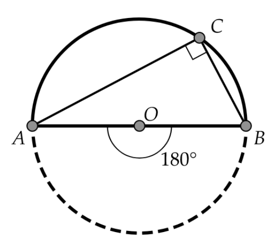


- Co-interior angles sum to 180°.

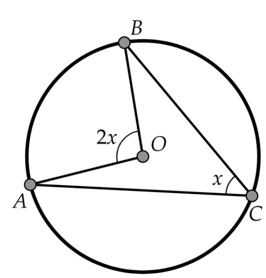


Circle Properties

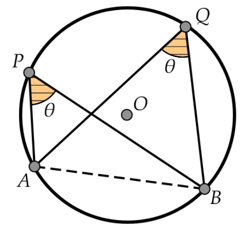
- The area of the sector formed by a central angle of $\theta°$ is given by $A_{\text{sector}} = \dfrac{\theta}{360}\pi r^2$, where r is the radius of the circle. The arc formed has length $s = \dfrac{\theta}{360}2\pi r$.
- The angle inscribed in a semicircle is a right angle.



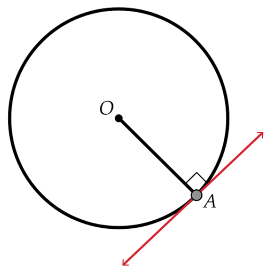
- When an arc subtends an inscribed angle and a central angle, the measure of the central angle is twice the measure of the inscribed angle.



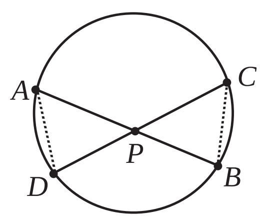
- Angles subtended by the same arc are equal.



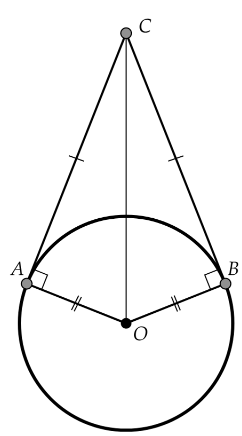
- A tangent to a circle and the radius drawn to the point of tangency meet at 90°.



- If two chords AB and CD of a circle intersect at the point P then $(PA)(PB) = (PC)(PD)$.

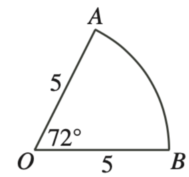


- Tangent segments from an external point to a circle are equal.



Sample Problems

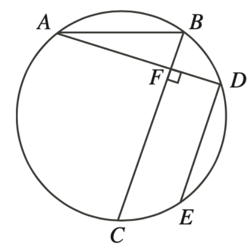
1. In the diagram, a sector of a circle with centre O, a radius of 5 and $\angle AOB = 72^\circ$ is shown. What is the perimeter of the sector?



Solution:

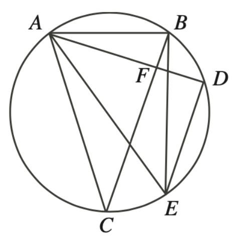
The perimeter of the sector is made up of two line segments (of total length $5 + 5 = 10$) and one arc of a circle. Since $\dfrac{72^\circ}{360^\circ} = \dfrac{1}{5}$, the length of the arc is $\dfrac{1}{5}$ of the total circumference of a circle of radius 5. Thus, the length of the arc is $\dfrac{1}{5}(2\pi(5)) = 2\pi$. Therefore, the perimeter of the sector is $10 + 2\pi$.

2. In the diagram, AB and BC are chords of the circle with $AB < BC$. If D is the point on the circle such that AD is perpendicular to BC and E is the point on the circle such that DE is parallel to BC, prove that $\angle EAC + \angle ABC = 90^\circ$.



Solution 1:

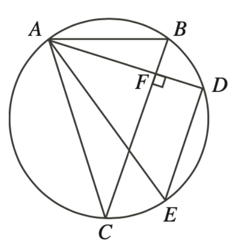
Join A to E and C, and B to E.



Since DE is parallel to BC and AD is perpendicular to BC, AD is perpendicular to DE, ie. $\angle ADE = 90^\circ$. Therefore, AE is a diameter. Now $\angle EAC = \angle EBC$, since both are subtended by EC. Therefore, $\angle EAC + \angle ABC = \angle EBC + \angle ABC = \angle EBA$ which is indeed equal to 90°, as required, since AE is a diameter.

Solution 2:

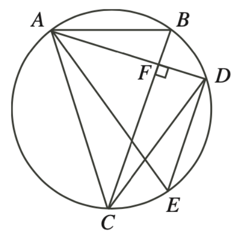
Join A to E and C.



Since DE is parallel to BC and AD is perpendicular to BC, AD is perpendicular to DE, ie. $\angle ADE = 90°$. Therefore, AE is a diameter. Thus, $\angle ECA = 90°$. Now $\angle ABC = \angle AEC$ since both are subtended by AC. Now $\angle EAC + \angle ABC = \angle EAC + \angle AEC = 180° - \angle ECA$ using the sum of the angles in $\triangle AEC$. But $\angle ECA = 90°$, so $\angle EAC + \angle AEC = 90°$.

Solution 3:

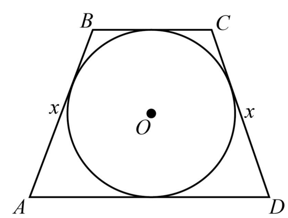
Join A to E and C, and C to D.



Since DE is parallel to BC and AD is perpendicular to BC, AD is perpendicular to DE, ie. $\angle ADE = 90°$. Therefore, AE is a diameter. Now $\angle ABC = \angle ADC$ since both are subtended by AC. Also $\angle EAC = \angle EDC$ since both are subtended by EC.

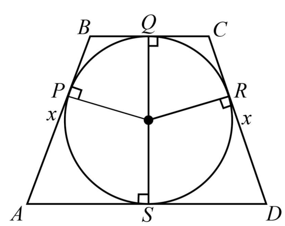
So $\angle EAC + \angle ABC = \angle EDC + \angle ADC = \angle ADE = 90°$.

3. In the isosceles trapezoid $ABCD$, $AB = CD = x$. The area of the trapezoid is 80. A circle with centre O is drawn inside the trapezoid such that it is tangent to all four sides of the trapezoid. Given that the radius of the circle is 4, determine the value of x.



Solution:

We label the points of tangency P, Q, R and S and connect them to the centre of the circle.



Each of the these line segments is a radius of the circle and so each is perpendicular to the sides of the trapezoid, since the sides of the trapezoid are tangent to the circle. The sides BC and AD are parallel and therefore, QO and OS are parallel and thus, QS is a diameter of the circle. Therefore, $QS = 2(4) = 8$ and so the height of the trapezoid is 8.

We let $BQ = b$ and $CQ = a$. Since tangent segments from an external point to a circle are equal, we obtain the following equalities.

$$BQ = BP = b$$
$$AP = AS = x - b$$
$$CQ = CR = a$$
$$DR = DS = x - a$$

Therefore,

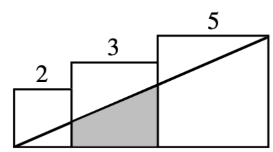
$$\begin{aligned} BC + AD &= BQ + QC + AS + SD \\ &= b + a + (x - b) + (x - a) \\ &= 2x \end{aligned}$$

The area of trapezoid $ABCD$ is equal to

$$\frac{1}{2}(BC + AD) \times QS = \frac{1}{2}(2x) \times 8 = 8x$$

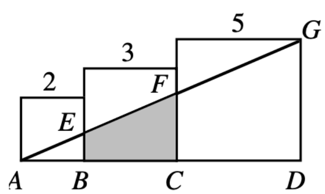
Since the area of the $ABCD$ is 80, we have $8x = 80$, which gives us that $x = 10$.

4. Three squares have dimensions as indicated in the diagram. What is the area of the shaded quadrilateral?



Solution

We make use of similar triangles. We start by labelling the diagram as shown.



We want to calculate the lengths EB and FC, which will allow us to calculate the area of $\triangle AEB$ and $\triangle AFC$. Since FC is parallel to GD, we have $\angle AFC = \angle AGD$ and $\angle ACF = \angle ADG$. In addition, we note that the angle at A is common to $\triangle AFC$ and $\triangle AGD$. Therefore, by angle-angle (AA), $\triangle AFC$ and $\triangle AGD$ are similar. So

$$\frac{AC}{AD} = \frac{FC}{GD} = \frac{5}{10} = \frac{1}{2}$$

Therefore, $FC = \frac{1}{2}GD = \frac{1}{2}(5) = \frac{5}{2}$.

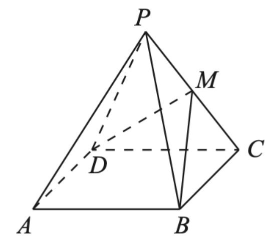
Using similar reasoning, $\triangle AEB$ and $\triangle AFC$ are also similar. Therefore,

$$\frac{AB}{AC} = \frac{EB}{FC} = \frac{2}{5}$$

Therefore, $EB = \frac{2}{5}FC = \frac{2}{5}\left(\frac{5}{2}\right) = 1$.

Thus, the shaded area is equal to the area of $\triangle AFC$ minus the area of $\triangle AEB$. Therefore, the area of the shaded quadrilateral is $\frac{1}{2}(5)\left(\frac{5}{2}\right) - \frac{1}{2}(2)(1) = \frac{21}{4}$. (Alternately, quadrilateral $EBCF$ is a trapezoid, where we know the base length is 3 and the two heights are 1 and $\frac{5}{2}$, and so the area is $\frac{1}{2}\left(1 + \frac{5}{2}\right)(3) = \frac{21}{4}$.)

5. In the diagram, $PABCD$ is a pyramid with square base $ABCD$ and with $PA = PB = PC = PD$. Suppose that M is the midpoint of PC and that $\angle BMD = 90^\circ$. Triangular-based pyramid $MBCD$ is removed by cutting along the triangle defined by the points M, B and D. The volume of the remaining solid $PABMD$ is 288. What is the length of AB?



Solution: *(This is a challenging problem.)*

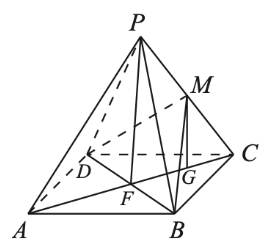
Let the side length of the square base $ABCD$ be $2a$ and the height of the pyramid (that is, the distance of P above the base) be $2h$.

Let F be the point of intersection of the diagonals AC and BD of the base. By symmetry, P is directly above F; that is, PF is perpendicular to the plane of square $ABCD$.

Note that $AB = BC = CD = DA = 2a$ and $PF = 2h$. We want to determine the value of $2a$.

Let G be the midpoint of FC.

Join P to F and M to G.



Consider $\triangle PCF$ and $\triangle MCG$. Since M is the midpoint of PC, we have $MC = \frac{1}{2}PC$. Since G is the midpoint of FC, we have $GC = \frac{1}{2}FC$.

Since $\triangle PCF$ and $\triangle MCG$ share an angle at C and the two pairs of corresponding sides adjacent to this angle are in the same ratio, $\triangle PCF$ is similar to $\triangle MCG$.

Since PF is perpendicular to FC, MG is perpendicular to GC.

Also, $MG = \frac{1}{2}PF = h$ since the side lengths of $\triangle MCG$ are half those of $\triangle PCF$.

The volume of the square-based pyramid $PABCD$ equals $\frac{1}{3}(AB^2)(PF) = \frac{1}{3}(2a)^2(2h) = \frac{8}{3}a^2h$.

Triangular-based pyramid $MBCD$ can be viewed as having right-angled $\triangle BCD$ as its base and MG as its height.

Thus, its volume equals $\frac{1}{3}\left(\frac{1}{2} \cdot BC \cdot CD\right)(MG) = \frac{1}{6}(2a)^2h = \frac{2}{3}a^2h$.

Therefore, the volume of solid $PABMD$, in terms of a and h, equals $\frac{8}{3}a^2h - \frac{2}{3}a^2h = 2a^2h$.

Since the volume of $PABMD$ is 288, we have $2a^2h = 288$ or $a^2h = 144$.

We have not yet used the information that $\angle BMD = 90^\circ$.

Since $\angle BMD = 90^\circ$, $\triangle BMD$ is right-angled at M and so $BD^2 = BM^2 + MD^2$.

By symmetry, $BM = MD$ and so $BD^2 = 2BM^2$.

Since $\triangle BCD$ is right-angled at C, we have $BD^2 = BC^2 + CD^2 = 2(2a)^2 = 8a^2$.

Since $\triangle BGM$ is right-angled at G, we have $BM^2 = BG^2 + MG^2 = BG^2 + h^2$.

Since $\triangle BFG$ is right-angled at F (the diagonals of square $ABCD$ are equal and perpendicular), it follows

$$\begin{aligned} BG^2 &= BF^2 + FG^2 \\ &= \left(\frac{1}{2}BD\right)^2 + \left(\frac{1}{4}AC\right)^2 \\ &= \frac{1}{4}BD^2 + \frac{1}{16}AC^2 \\ &= \frac{1}{4}BD^2 + \frac{1}{16}BD^2 \\ &= \frac{5}{16}BD^2 \\ &= \frac{5}{2}a^2 \end{aligned}$$

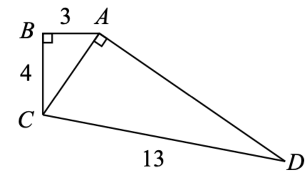
Since $2BM^2 = BD^2$, it follows that $2(BG^2 + h^2) = 8a^2$ which gives $\frac{5}{2}a^2 + h^2 = 4a^2$ or $h^2 = \frac{3}{2}a^2$ or $a^2 = \frac{2}{3}h^2$.

Since $a^2h = 144$, we have $\frac{2}{3}h^2 \cdot h = 144$ or $h^3 = 216$ which gives $h = 6$. From $a^2h = 144$, we obtain $6a^2 = 144$ or $a^2 = 24$.

Since $a > 0$, it follows that $a = 2\sqrt{6}$ and so $AB = 2a = 4\sqrt{6}$.

Problem Set

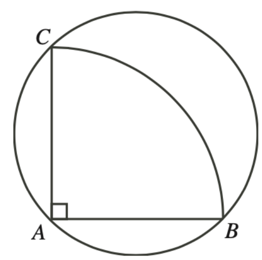
1. In the diagram, $\triangle ABC$ is right-angled at B and $\triangle ACD$ is right-angled at A. Also, $AB = 3, BC = 4$, and $CD = 13$. What is the area of quadrilateral $ABCD$?



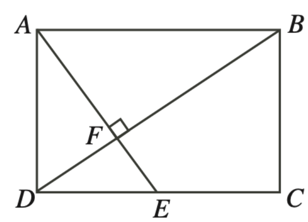
2. One of the faces of a rectangular prism has area 27 cm^2. Another face has area 32 cm^2. If the volume of the prism is 144 cm^3, determine the surface area of the prism in cm^2.

3. The sum of the radii of two circles is 10 cm. The circumference of the larger circle is 3 cm greater than the circumference of the smaller circle. Determine the difference between the area of the larger circle and the area of the smaller circle.

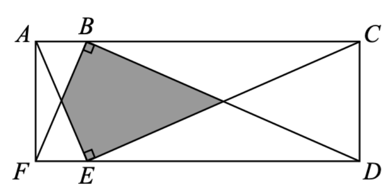
4. In the diagram, ABC is a quarter of a circular pizza with centre A and radius 20 cm. The piece of pizza is placed on a circular pan with A, B and C touching the circumference of the pan, as shown. What fraction of the pan is covered by the piece of pizza?



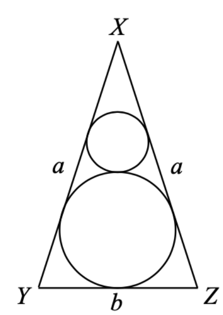
5. In rectangle $ABCD$, point E is on side DC. Line segments AE and BD are perpendicular and intersect at F. If $AF = 4$ and $DF = 2$, determine the area of quadrilateral $BCEF$.



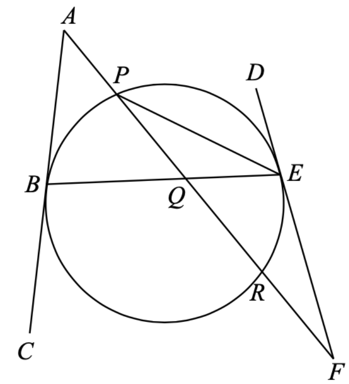
6. In the diagram, $ACDF$ is a rectangle B with $AC = 200$ and $CD = 50$. Also, $\triangle FBD$ and $\triangle AEC$ are congruent triangles which are right-angled at B and E, respectively. What is the area of the shaded region?



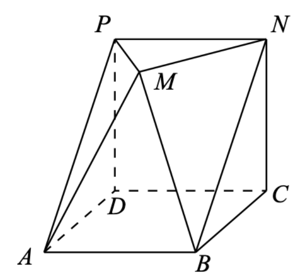
7. In the diagram, $\triangle XYZ$ is isosceles with $XY = XZ = a$ and $YZ = b$ where $b < 2a$. A larger circle of radius R is inscribed in the triangle (that is, the circle is drawn so that it touches all three sides of the triangle). A smaller circle of radius r is drawn so that it touches XY, XZ and the larger circle. Determine an expression for $\dfrac{R}{r}$ in terms of a and b.



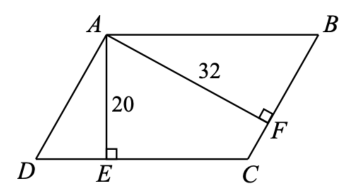
8. In the diagram, line segments AC and DF are tangent to the circle at B and E, respectively. Also, AF intersects the circle at P and R, and intersects BE at Q, as shown. If $\angle CAF = 35°$, $\angle DFA = 30°$, and $\angle FPE = 25°$, determine the measure of $\angle PEQ$.



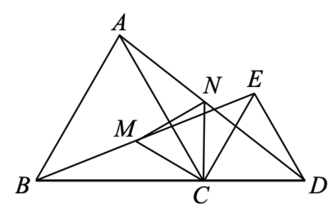
9. In the diagram, $ABCD$ and $PNCD$ are squares of side length 2, and $PNCD$ is perpendicular to $ABCD$. Point M is chosen on the same side of $PNCD$ as AB so that $\triangle PMN$ is parallel to $ABCD$, so that $\angle PMN = 90°$, and so that $PM = MN$. Determine the volume of the convex solid $ABCDPMN$.



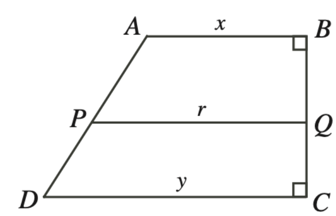
10. In the diagram, $ABCD$ is a parallelogram. Point E is on DC with AE perpendicular to DC, and point F is on CB with AF perpendicular to CB. If $AE = 20$, $AF = 32$, and $\cos(\angle EAF) = \frac{1}{3}$, determine the exact value of the area of quadrilateral $AECF$.



11. In the diagram, C lies on BD. Also, $\triangle ABC$ and $\triangle ECD$ are equilateral triangles. If M is the midpoint of BE and N is the midpoint of AD, prove that $\triangle MNC$ is equilateral.



12. $ABCD$ is a trapezoid with parallel sides AB and DC. Also, BC is perpendicular to AB and to DC. The line PQ is parallel to AB and divides the trapezoid into two regions of equal area. If $AB = x$, $DC = y$, and $PQ = r$, prove that $x^2 + y^2 = 2r^2$.



13. Square $WXYZ$ has side length 6 and is drawn, as shown, completely inside a larger square $EFGH$ with side length 10, so that the squares do not touch and so that WX is parallel to EF. Prove that the sum of the areas of trapezoid $EFXW$ and trapezoid $GHZY$ does not depend on the position of $WXYZ$ inside $EFGH$.