Analytic Geometry
Solutions

1. Using the line segment from $O(0,0)$ to $C(9,0)$ as the base and noting that the height is 4, the area of triangle OCD is 18. We let the vertical line be $x = k$. The line from $O(0,0)$ to $D(8,4)$ is $y = \frac{1}{2}x$ and this line intersects the vertical line at $K\left(k, \frac{1}{2k}\right)$. Let $L = (k,0)$ be the x-intercept of the vertical line. The area of triangle OKL must be $\frac{18}{2} = 9$. Therefore, $\frac{1}{4}k^2 = 9$ and so the vertical line required is $x = 6$. (The value of $k = -6$ is not admissible since the line $x = -6$ does not intersect the triangle.)

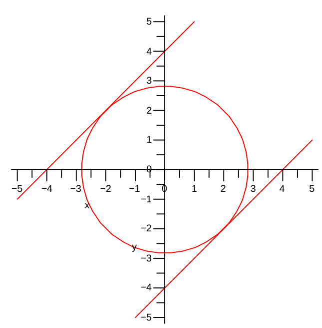
2. There are several ways to solve this question. We will give two solutions using analytic geometry.

   Solution 1

   If the line is tangent to the circle, then the distance from the centre (0,0) to the line $y = x + c$ (or $x - y + c = 0$) equals the radius of the circle which is $2\sqrt{2}$. Using the formula for distance from a point to a line,
   $$2\sqrt{2} = \frac{|c|}{\sqrt{2}}$$
   Therefore, we have $c = \pm 4$.

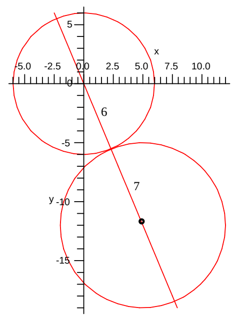
   Solution 2

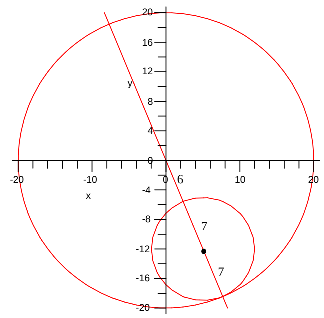
   We substitute $y = x + c$ into the equation of the circle to obtain $x^2 + (x + c)^2 = 8$. Expanding we obtain $2x^2 + 2xc + c^2 - 8 = 0$. If the line is tangent to the circle, then we need to find values of c such that this quadratic has exactly one root. Therefore, it must be the case that
   $$\begin{aligned} 4c^2 - 4(2)(c^2 - 8) &= 0 \\ 4c^2 - 8c^2 - 64 &= 0 \\ 4c^2 &= 64 \\ c^2 &= 16 \\ c &= \pm 4 \end{aligned}$$



3. There are two circles, the first with its centre at (0,0) and with radius $|k|$, and the second with its centre at (5,−12) and with radius 7. The distance between the centres can be calculated to

be $\sqrt{(-5)^2+(12)^2} = 13$. Since the two circles intersect only once, they can be either externally or internally tangent. If they are externally tangent, $|k| + 7 = 13$ and so $k = 6$ or $k = -6$. If they are internally tangent, $|k| - 7 = 13$ and so $k = 20$ or -20. So we have 4 possible values of k: $-6, 6, -20$ and 20.





4. **Solution 1**

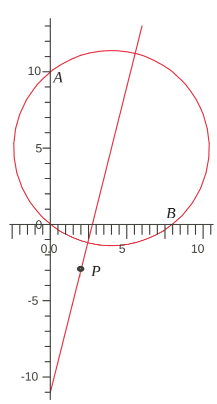
All lines that cut a circle in half pass through the centre. Now the perpendicular bisector of any chord passes through the centre. If we consider the vertical chord from $(0, 0)$ to $(0, 10)$, the perpendicular bisector is the horizontal line $y = 5$. Similarly, if we consider the horizontal chord from $(0, 0)$ to $(8, 0)$, the perpendicular bisector is the vertical line $x = 4$. These two lines intersect at $(4, 5)$ and therefore, the centre is $(4, 5)$. We require the y-intercept of the line through $(4, 5)$ and $P(2, -3)$. This line is $y = 4x - 11$ and the y-intercept is -11.

Solution 2

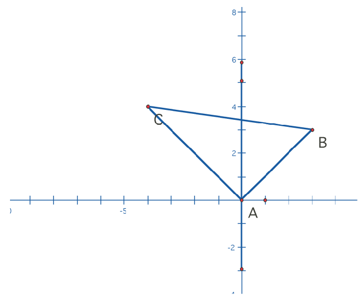
Observe that $\triangle AOB$ is right-angled at O, thus AB is the diameter of the circle, and its midpoint $(4, 5)$ is the centre of the circle. As in solution 1, we require the y-intercept of the line that goes through $(4, 5)$ and $P(2, -3)$. This line is $y = 4x - 11$ and the y-intercept is -11.

Solution 3

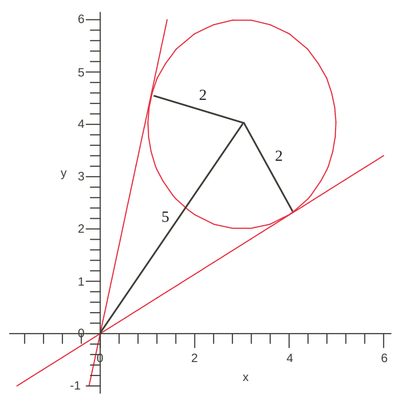
The general equation of a circle with centre (h, k) and radius r is $(x - h)^2 - (y - k)^2 = r^2$. Substituting in the point $(0, 0)$ we obtain the equation $h^2 + k^2 = r^2$. Substituting in the point $(0, 10)$ we obtain the equation $h^2 + (10 - k)^2 = r^2$. Therefore, $(10 - k)^2 = k^2$ which gives $k = 5$. Substituting the point $(8, 0)$ into the equation of the circle gives $(8 - h)^2 + k^2 = r^2$. Therefore, $(8 - h)^2 = h^2$ which gives $h = 4$. As in solution 1, we require the y-intercept of the line that goes through $(4, 5)$ and $P(2, -3)$. This line is $y = 4x - 11$ and the y-intercept is -11.



5. Since the slopes of AB and AC are 1 and -1 respectively, the required line is vertical and its equation is $x = 0$.



6. Consider a triangle with the following vertices: the origin, the centre of the circle and a point of tangency of one of the two tangents we are considering. Since there are two tangents, we have two such triangles. A tangent is perpendicular to the radius at the point of tangency and so these two triangles are right-angled. The two known sides of one of these right triangles are the radius 2 and the segment from (0,0) to (3,4) which has length 5. Thus, the other side has length $\sqrt{21}$.



Since each of the tangents pass through the origin, their equations are of the form $y = mx$. We are interested in values of m for which the line $y = mx$ intersects the circle only once. Substituting into the equation of the circle we get

$$\begin{aligned} (x-3)^2 + (mx-4)^2 &= 4 \\ x^2 - 6x + 9 + m^2x^2 - 8mx + 16 &= 4 \\ (1+m^2)x^2 - (6+8m)x + 21 &= 0 \end{aligned}$$

Now this quadratic will have one solution when its discriminant is zero. Thus, we are looking for values of m that give a discriminant of 0. So

$$\begin{aligned} (6+8m)^2 - 4 \cdot 21 \cdot (1+m^2) &= 0 \\ 36 + 96m + 64m^2 - 84 - 84m^2 &= 0 \\ -20m^2 + 96m - 48 &= 0 \\ m &= \frac{12 \pm 2\sqrt{21}}{5} \end{aligned}$$

Therefore, both tangents we are considering have length $\sqrt{21}$ and their slopes are $\dfrac{12 \pm 2\sqrt{21}}{5}$.

7. The required set of points is the line that is the perpendicular bisector of the line segment CD. Since CD has slope $-\dfrac{1}{2}$ and midpoint $M = \left(3, \dfrac{3}{2}\right)$, the required line passes through M and has slope 2. The equation of the resulting line is $4x - 2y - 9 = 0$.

8. We present the solution that uses analytic geometry most directly. Let the coordinates of the points be $K(0,0)$, $W(x,y)$, $A(a,b)$ and $D(d,0)$. Therefore, the coordinates of M and N are $M\left(\dfrac{x}{2}, \dfrac{y}{2}\right)$ and $N\left(\dfrac{a+d}{2}, \dfrac{b}{2}\right)$. Now we are given that $2MN = AW + DK$. Therefore,

$$2\sqrt{\left(\frac{a+d-x}{2}\right)^2 + \left(\frac{b-y}{2}\right)^2} = \sqrt{(a-x)^2 + (b-y)^2} + d$$

Squaring both sides and simplifying (using the fact that $d \neq 0$) gives

$$\begin{aligned}(a+d-x)^2 + (b-y)^2 &= (a-x)^2 + (b-y)^2 + 2d\sqrt{(a-x)^2 + (b-y)^2} + d^2 \\ 2d(a-x) &= 2d\sqrt{(a-x)^2 + (b-y)^2} \\ (a-x) &= \sqrt{(a-x)^2 + (b-y)^2}\end{aligned}$$

Squaring both sides again and simplifying gives

$$\begin{aligned}(a-x)^2 &= (a-x)^2 + (b-y)^2 \\ (b-y)^2 &= 0\end{aligned}$$

This result gives $b = y$ and implies that the slope of AW is 0. Therefore, AW is parallel to KD.

9. Let the coordinates of A and B be (a, c) and (b, d), respectively. Point A satisfies the equation of the first line and point B satisfies the equation of the second line and so $4a+3c-48 = 0$ and $b+3d+10 = 0$. Moreover, since (4,2) is the midpoint, we know $\dfrac{a+b}{2} = 4$ and $\dfrac{c+d}{2} = 2$. Thus, $b = 8 - a$ and $d = 4 - c$. Substituting these into the second equation above and simplifying we obtain $-a - 3c + 30 = 0$. Adding this equation to the first equation gives $3a - 18 = 0$ and so $a = 6$. From the first equation we obtain that $c = 8$. Therefore, $b = 2$ and $d = -4$. So the coordinates of A and B are $(6, 8)$ and $B(2, -4)$, respectively.