Analytic Geometry

Toolkit

Along with the material on parabolas in the workshop on Functions, Equations and Polynomials, some useful formulae include:

Description	Formula
Slope of a line through points (x_1, y_1) and (x_2, y_2)	$\frac{y_2 - y_1}{x_2 - x_1}$
Two perpendicular lines with slopes m_1 and m_2	$m_2 = -\frac{1}{m_1}$
Standard form of an equation of a line with slope $-\frac{A}{B}$, x-intercept $-\frac{C}{A}$, and y-intercept $-\frac{C}{B}$	$Ax + By + C = 0$
Equation of a line with slope m through the point (x_0,y_0)	$y - y_0 = m(x - x_0)$
Equation of a line with intercepts at $(a, 0)$ and $(0, b)$	$\frac{x}{a} + \frac{y}{b} = 1$
Formula for the midpoint of $A(x_1,y_1)$ and $B(x_2,y_2)$	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Distance D between points $A(x_1,y_1)$ and $B(x_2,y_2)$	$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
(Minimum) distance D between the line $Ax + By + C = 0$ and the point (x_0,y_0)	$D = \frac{\lvert Ax_0 + By_0 + C\rvert}{\sqrt{A^2 + B^2}}$
Area of a triangle with vertices $A(x_1,y_1)$, $B(x_2,y_2)$, and $C(x_3,y_3)$	$\frac{1}{2}\lvert x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3\rvert$
Equation of a circle centred at (h,k) with radius r	$(x - h)^2 + (y - k)^2 = r^2$

Sample Problems

1. If the line $2x - 3y - 6 = 0$ is reflected in the line $y = -x$, find the equation of the reflected line.

   Solution

   Recall that after reflection in a line, the distance from the image of any point to the line of reflection is the same as the distance from the original point to the line. Thus, the line segment joining any point to its image is perpendicular to the line of reflection and the midpoint of this line segment is on the line of reflection.

   The line $2x - 3y - 6 = 0$ has intercepts of $(3, 0)$ and $(0, -2)$. Since the image of a line after reflection is another line, we reflect these two points and then find the equation of the line through their image points. The image of the point $(3, 0)$ upon reflection in the line $y = -x$ is $(0, -3)$. This result follows since the line segment from $(3, 0)$ to $(0, -3)$ has slope 1, which makes it perpendicular to $y = -x$, and its midpoint is $\left(\frac{3}{2}, -\frac{3}{2}\right)$, which is on the line $y = -x$. Similarly, the image of the point $(0, -2)$ reflected in the line $y = -x$ is (2,0). Since the line passes through $(0, -3)$ and $(2, 0)$, the slope of the line is $\frac{3}{2}$ and the equation of the line is $y = \frac{3}{2}x - 3$ or $3x - 2y - 6 = 0$.

2. If $A(3, 5)$ and $B(11, 11)$ are fixed points, find the point(s) P on the x-axis such that the area of the triangle ABP equals 30.

   Solution 1

   The length of AB is $\sqrt{(11-3)^2 + (11-5)^2} = \sqrt{8^2 + 6^2} = 10$. The slope of the line through A and B is $\frac{11-5}{11-3} = \frac{6}{8} = \frac{3}{4}$. Therefore, the equation of the line is $y - 11 = \frac{3}{4}(x - 11)$ or $3x - 4y + 11 = 0$. Now if we think of AB as the base of the triangle, then the distance from $P(a,0)$ to the line AB must be the height of the triangle and therefore, this distance is $\frac{30(2)}{10} = 6$. Thus,

   $$\begin{aligned} \frac{|3a - 4(0) + 11|}{\sqrt{3^2 + (-4)^2}} &= 6 \\ 3a + 11 &= \pm 30 \\ a &= -\frac{41}{3} \text{ or } \frac{19}{3} \end{aligned}$$

   The points are $\left(\frac{19}{3}, 0\right)$ and $\left(-\frac{41}{3}, 0\right)$.

   Solution 2

   Let $P = (p, 0)$. Then using the formula for the area of a triangle given its vertices we obtain

   $$\begin{aligned} \text{area of } \triangle ABP &= \frac{1}{2}|x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3| \\ &= \frac{1}{2}|33 + 0 + 5p - 55 - 11p - 0| \\ &= |-22 - 6p| \end{aligned}$$

Therefore, $|-22-6p| = 60$ and so $p = \dfrac{19}{3}$ or $p = -\dfrac{41}{3}$.

The points are $\left(\dfrac{19}{3}, 0\right)$ and $\left(-\dfrac{41}{3}, 0\right)$

3. Given two circles, the line joining their points of intersection is called their *common chord*. It can be shown that the common chord is perpendicular to the line connecting the centres of the circles. (Can you prove this?) Given the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x + 2 = 0$, find the length of their common chord.

   Solution

   The first circle has centre (0,0) and radius 2. By completing the square, the equation for the second circle can be written as
   $$(x-3)^2 + y^2 = 7$$
   and therefore, it has centre (3,0) and radius $\sqrt{7}$. Since the line joining the centres is horizontal, the common chord is vertical. We can find the intersection points of the two circles by solving
   $$\begin{aligned} x^2 + y^2 - 4 &= x^2 + y^2 - 6x + 2 \\ 6x &= 6 \\ x &= 1 \end{aligned}$$
   Substituting this value back into the equation for either circle gives intersection points $(1, \pm\sqrt{3})$. Thus, the length of the common chord is $2\sqrt{3}$.

4. A line has slope -2 and is a distance of 2 units from the origin. What is the area of the triangle formed by this line and the axes?

   Solution

   Let the x-intercept of the line be k. Since the line has slope -2, the y-intercept is $2k$, and the equation of the line is $2x + y - 2k = 0$. Therefore, the formula for the distance from a line to a point shows that the distance from this line to (0,0) is $\left|\dfrac{2k}{\sqrt{5}}\right|$. Since this distance is 2, we have that $k = \pm\sqrt{5}$. The base of the triangle is the distance from the origin to the x-intercept. This distance is $|k| = \sqrt{5}$. The height of the triangle is the distance from the origin to the y-intercept. This distance is $|2k| = 2\sqrt{5}$. Therefore, the area of the required triangle is $\dfrac{1}{2} \cdot \sqrt{5} \cdot 2\sqrt{5} = 5$.

Problem Set

1. A vertical line divides the triangle with vertices $O(0,0)$, $C(9,0)$, and $D(8,4)$ into two regions of equal area. Find the equation of the line.

2. Find all value(s) of c such that the line $y = x + c$ is tangent to the circle $x^2 + y^2 = 8$.

3. Find all value(s) of k so that the circle with equation $x^2 + y^2 = k^2$ will intersect the circle with equation $(x - 5)^2 + (y + 12)^2 = 49$ in exactly one point.

4. A circle intersects the axes at A(0,10), O(0,0) and B(8,0). A line through P(2,-3) cuts the circle in half. What is the y intercept of the line?

5. If triangle ABC has vertices A(0,0), B(3,3), and C(-4,4), determine the equation of the bisector of $\angle CAB$.

6. What are the length(s) and the slope(s) of the tangent(s) from the origin to the point of tangency with the circle $(x - 3)^2 + (y - 4)^2 = 4$?

7. Find the equation of the set of points equidistant from $C(0,3)$ and $D(6,0)$.

8. In quadrilateral $KWAD$, K is at the origin, D is on the positive x-axis and A and W are in the first quadrant. The midpoints of KW and AD are M and N, respectively. Given that $MN = \dfrac{1}{2}(AW + DK)$, prove that AW is parallel to KD.

9. The point A is on the line $4x + 3y - 48 = 0$ and the point B is on the line $x + 3y + 10 = 0$. If the midpoint of AB is (4,2), find the co-ordinates of A and B.