The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Pascal Contest

(Grade 9)

Wednesday, February 28, 2024
(in North America and South America)

Thursday, February 29, 2024
(outside of North America and South America)

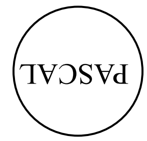
Solutions

1. Calculating, $2 - 0 + 2 - 4 = 2 + 2 - 4 = 0$.

   Answer: (B)

2. The distance between two numbers on the number line is equal to their positive difference. Here, this distance is $6 - (-5) = 11$.

   Answer: (D)

3. Since a turn of 180° is a half-turn, the resulting figure is  . (Note that we would obtain the same result rotating by 180° clockwise or 180° counterclockwise.)

   Answer: (C)

4. Since July 1 is a Wednesday, then July 8 and July 15 are both Wednesdays.
   Since July 15 is a Wednesday, then July 17 is a Friday.

   Answer: (D)

5. The first rhombus and the last rhombus each have three edges that form part of the exterior of the figure, and so they each contribute 3 to the perimeter.
   The inner four rhombi each have two edges that form part of the exterior of the figure, and so they each contribute 2 to the perimeter.
   Thus, the perimeter is $2 \times 3 + 4 \times 2 = 14$.

   Answer: (B)

6. On Monday, Narsa ate 4 cookies.
   On Tuesday, Narsa ate 12 cookies.
   On Wednesday, Narsa ate 8 cookies.
   On Thursday, Narsa ate 0 cookies.
   On Friday, Narsa ate 6 cookies.
   This means that Narsa ate $4 + 12 + 8 + 0 + 6 = 30$ cookies.
   Since the package started with 45 cookies, there are $45 - 30 = 15$ cookies left in the package after Friday.

   Answer: (D)

7. For there to be equal numbers of each colour of candy, there must be at most 3 red candies and at most 3 yellow candies, since there are 3 blue candies to start.
   Thus, Shuxin ate at least 7 red candies and at least 4 yellow candies.
   This means that Shuxin ate at least $7 + 4 = 11$ candies.
   We note that if Shuxin eats 7 red candies, 4 yellow candies, and 0 blue candies, there will indeed be equal numbers of each colour.

   Answer: (C)

8. Since 10 students have black hair and 3 students have black hair and wear glasses, then a total of $10 - 3 = 7$ students have black hair but do not wear glasses.

   Answer: (A)

9. Since 25% is equivalent to $\frac{1}{4}$, then the fraction of the trail covered by the section along the river and the section through the forest is $\frac{1}{4} + \frac{5}{8} = \frac{2}{8} + \frac{5}{8} = \frac{7}{8}$.
   This means that the final section up a hill represents $1 - \frac{7}{8} = \frac{1}{8}$ of the trail.
   Since $\frac{1}{8}$ of the trail is 3 km long, then the entire trail is 8×3 km $= 24$ km long.

   Answer: (A)

10. Using the definition, $(5\nabla 2)\nabla 2 = (4 \times 5 + 2)\nabla 2 = 22\nabla 2 = 4 \times 22 + 2 = 90$.

    Answer: (E)

11. *Solution 1*
    If all of Lauren's 10 baskets are worth 2 points, she would have $10 \times 2 = 20$ points in total.
    Since she has 26 points in total, then she scores $26 - 20 = 6$ more points than if all of her baskets are worth 2 points.
    This means that, if 6 of her baskets are worth 3 points, she would gain 1 point for each of these 6 baskets and so have $20 + 6 = 26$ points.
    Thus, she makes 6 baskets worth 3 points.
    (We note that $6 \times 3 + 4 \times 2 = 26$.)

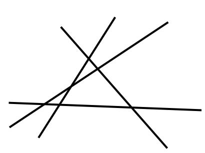
    Solution
    Suppose that Lauren makes x baskets worth 3 points each.
    Since she makes 10 baskets, then $10 - x$ baskets that she made are worth 2 points each.
    Since Lauren scores 26 points, then $3x + 2(10 - x) = 26$ and so $3x + 20 - x = 26$ which gives $x = 6$.
    Therefore, Lauren makes 6 baskets worth 3 points.

    Answer: (B)

12. From the given list, the numbers 11 and 13 are the only prime numbers, and so must be Karla's and Levi's numbers in some order.
    From the given list, 16 is the only perfect square; thus, Glen's number was 16.
    The remaining numbers are 12, 14, 15.
    Since Hao's and Julia's numbers were even, then their numbers must be 12 and 14 in some order.
    Thus, Ioana's number is 15.

    Answer: (B)

13. Each of the 4 lines can intersect each of the other 3 lines at most once.
    This might appear to create $4 \times 3 = 12$ points of intersection, but each point of intersection is counted twice – one for each of the 2 lines.
    Thus, the maximum number of intersection points is $\dfrac{4 \times 3}{2} = 6$.
    The diagram below demonstrates that 6 intersection points are indeed possible:



    Answer: (D)

14. When 10 numbers have an average of 17, their sum is $10 \times 17 = 170$.
    When 9 numbers have an average of 16, their sum is $9 \times 16 = 144$.
    Therefore, the number that was removed was $170 - 144 = 26$.

    Answer: (A)

15. Since $CD = DE = EC$, then $\triangle CDE$ is equilateral, which means that $\angle DEC = 60^\circ$.
Since $\angle DEB$ is a straight angle, then $\angle CEB = 180^\circ - \angle DEC = 180^\circ - 60^\circ = 120^\circ$.
Since $CE = EB$, then $\triangle CEB$ is isosceles with $\angle ECB = \angle EBC$.
Since $\angle ECB + \angle CEB + \angle EBC = 180^\circ$, then $2 \times \angle EBC + 120^\circ = 180^\circ$, which means that $2 \times \angle EBC = 60^\circ$ or $\angle EBC = 30^\circ$.
Therefore, $\angle ABC = \angle EBC = 30^\circ$.

ANSWER: (A)

16. Since $x^2 < x$ and $x^2 \geq 0$, then $x > 0$ and so it cannot be the case that x is negative.
Thus, neither (D) nor (E) is the answer.
Since $x^2 < x$, then we cannot have $x > 1$. This is because when $x > 1$, we have $x^2 > x$.
Thus, (A) is not the answer and so the answer is (B) or (C).
If $x = \frac{1}{3}$, then $x^2 = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$ and $\frac{x}{2} = \frac{1/3}{2} = \frac{1}{6}$.
Since $\frac{1}{6} > \frac{1}{9}$, then (B) cannot be the answer.
Therefore, the answer must be (C).
Checking, when $x = \frac{3}{4}$, we have $x^2 = \frac{9}{16}$ and $\frac{x}{2} = \frac{3}{8}$.
Since $\frac{x}{2} = \frac{3}{8} = \frac{6}{16} < \frac{9}{16} = x^2$, then $\frac{x}{2} < x^2$.
Also, $x^2 = \frac{9}{16} < \frac{12}{16} = \frac{3}{4} = x$.
This confirms that $x = \frac{3}{4}$ does satisfy the required conditions.

ANSWER: (C)

17. In 2 hours travelling at 100 km/h, Melanie travels 2 h $\times$ 100 km/h $=$ 200 km.
When Melanie travels 200 km at 80 km/h, it takes $\frac{200 \text{ km}}{80 \text{ km/h}} = 2.5$ h.
Melanie travels a total of 200 km $+$ 200 km $=$ 400 km.
Melanie travels for a total of 2 h $+$ 2.5 h $=$ 4.5 h.
Therefore, Melanie's average speed is $\frac{400 \text{ km}}{4.5 \text{ h}} \approx 88.89$ km/h.
Of the given choices, this is closest to 89 km/h.

ANSWER: (B)

18. From the given information, we know that

$$\mathrm{S} + \mathrm{E} + \mathrm{T} = 2 \qquad \mathrm{H} + \mathrm{A} + \mathrm{T} = 7 \qquad \mathrm{T} + \mathrm{A} + \mathrm{S} + \mathrm{T} + \mathrm{E} = 3 \qquad \mathrm{M} + \mathrm{A} + \mathrm{T} = 4$$

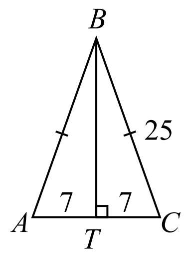
Since $\mathrm{T} + \mathrm{A} + \mathrm{S} + \mathrm{T} + \mathrm{E} = 3$ and $\mathrm{S} + \mathrm{E} + \mathrm{T} = 2$, then $\mathrm{T} + \mathrm{A} = 3 - 2 = 1$.
Since $\mathrm{H} + \mathrm{A} + \mathrm{T} = 7$ and $\mathrm{T} + \mathrm{A} = 1$, then $\mathrm{H} = 7 - 1 = 6$.
Since $\mathrm{M} + \mathrm{A} + \mathrm{T} = 4$ and $\mathrm{H} = 7$, then $\mathrm{M} + (\mathrm{A} + \mathrm{T}) + \mathrm{H} = 4 + 6 = 10$.
Therefore, the value of the word MATH is 10.
We note that it is also possible to find specific values for S, E, T, A that give the correct values to the words. One such set of values is $\mathrm{A} = 1$, $\mathrm{T} = 0$, $\mathrm{S} = 4$, and $\mathrm{E} = -2$. These values are not unique, even though the values assigned to M and H (namely, 3 and 6) are unique.

ANSWER: (E)

19. The perimeter of $\triangle ABC$ is equal to $(3x+4)+(3x+4)+2x = 8x+8$.
The perimeter of rectangle $DEFG$ is equal to

$$2 \times (2x-2) + 2 \times (3x-1) = 4x - 4 + 6x - 2 = 10x - 6$$

Since these perimeters are equal, we have $10x - 6 = 8x + 8$ which gives $2x = 14$ and so $x = 7$.
Thus, $\triangle ABC$ has $AC = 2 \times 7 = 14$ and $AB = BC = 3 \times 7 + 425$.
We drop a perpendicular from B to T on AC.



Since $\triangle ABC$ is isosceles, then T is the midpoint of AC, which gives $AT = TC = 7$.
By the Pythagorean Theorem, $BT = \sqrt{BC^2 - TC^2} = \sqrt{25^2 - 7^2} = \sqrt{625 - 49} = \sqrt{576} = 24$.
Therefore, the area of $\triangle ABC$ is equal to $\frac{1}{2} \cdot AC \cdot BT = \frac{1}{2} \times 14 \times 24 = 168$.

Answer: (C)

20. Since N is between 1 000 000 and 10 000 000, inclusive, then $25 \times N$ is between 25 000 000 and 250 000 000, inclusive, and so $25 \times N$ has 8 digits or it has 9 digits.
We consider the value of $25 \times N$ as having 9 digits, with the possibility that the first digit could be 0.
Since $25 \times N$ is a multiple of 25, its final two digits must be 00, 25, 50, or 75.
For a fixed set of leftmost three digits, xyz, the multiple of 25 that has the largest sum of digits must be $xyz\,999\,975$ since the next four digits are as large as possible (all 9s) and the rightmost two digits have the largest possible sum among the possible endings for multiples of 25.
So to answer the question, we need to find the integer of the form $xyz\,999\,975$ which is between 25 000 000 and 250 000 000 and has the maximum possible sum $x + y + z$.
We know that the maximum possible value of x is 2, the maximum possible value of y is 9, and the maximum possible value of z is 9.
This means that $x + y + z \leq 2 + 9 + 9 = 20$.
We cannot have 299 999 975 since it is not in the given range.
However, we could have $x + y + z = 19$ if $x = 1$ and $y = 9$ and $z = 9$.
Therefore, the integer 199 999 975 is the multiple of 25 in the given range whose sum of digits is as large as possible. This sum is $1 + 6 \times 9 + 7 + 5 = 67$.
We note that $199\,999\,975 = 25 \times 7\,999\,999$ so it is a multiple of 25. Note that $N = 7\,999\,999$ is between 1 000 000 and 10 000 000.

Answer: (C)

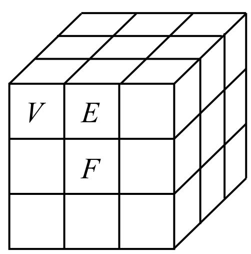
21. Since the second column includes the number 1, then step (ii) was never used on the second column, otherwise each entry would be at least 2.
To generate the 1, 3 and 2 in the second column, we thus need to have used step (i) 1 time on row 1, 3 times on row 2, and 2 times on row 3.
This gives:

1	1	1
3	3	3
2	2	2

We cannot use step (i) any more times, otherwise the entries in column 2 will increase. Thus, $a = 1 + 3 + 2 = 6$.
To obtain the final grid from this current grid using only step (ii), we must increase each entry in column 1 by 6 (which means using step (ii) 3 times) and increase each entry in column 3 by 4 (which means using step (ii) 2 times). Thus, $b = 3 + 2 = 5$.
Therefore, $a + b = 11$.

Answer: 11

22. The 27 small cubes that make up the larger $3 \times 3 \times 3$ can be broken into 4 categories: 1 small cube in the very centre of the larger cube (not seen in the diagram), 8 small cubes at the vertices of larger cube (an example is marked with V), 12 small cubes on the edges not at vertices (an example is marked with E), and 6 small cubes at the centre of each face (an example is marked with F).



The centre cube contributes 0 to the surface area of the cube.
Each of the 8 vertex cubes contributes 3 to the surface area of the larger cube, as 3 of the 6 faces of each such cube are on the exterior of the larger cube.
Each of the 12 edge cubes contributes 2 to the surface area of the larger cube.
Each of the 6 face cubes contributes 1 to the surface area of the larger cube.
There are 10 small red cubes that need to be placed as part of the larger cube.
To minimize the surface area that is red, we place the red cubes in positions where they will contribute the least to the overall surface area. To do this, we place 1 red cube at the centre (contributing 0 to the surface area), 6 red cubes at the centres of the faces (each contributing 1 to the surface area), and the remaining 3 red cubes on the edges (each contributing 2 to the surface area).
In total, the surface area that is red is $1 \times 0 + 6 \times 1 + 3 \times 2 = 12$.

Answer: 12

23. We want to count the number of four-digit codes $abcd$ that satisfy the given rules.
From the first rule, at least one of the digits must be 4, but $b \neq 4$ and $d \neq 4$.
Therefore, either $a = 4$ or $c = 4$. The fourth rule tells us that we could have both $a = 4$ and $c = 4$.

Suppose that $a = 4$ and $c = 4$.
The code thus has the form $4b4d$.
The second and third rules tell us that the remaining digits are 2 and 7, and that there are no further restrictions on where the 2 and 7 are placed.
Therefore, in this case, the code is either 4247 or 4742, and so there are 2 possible codes.

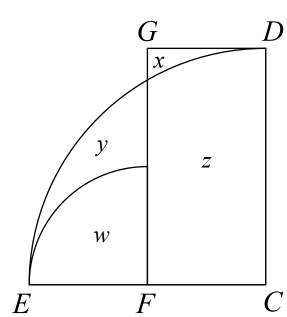
Suppose that $a = 4$ and $c \neq 4$. (Recall that $b \neq 4$ and $d \neq 4$.)
The code thus has the form $4bcd$.
The remaining digits include a 2 (which can be placed in any of the remaining positions), a 7, and either a 1 or a 6.
There are 3 positions in which the 2 can be placed, after which there are 2 positions in which the 7 can be placed, after which there are 2 digits that can be placed in the remaining position.
Therefore, in this case, there are $3 \times 2 \times 2 = 12$ possible codes.

Suppose that $c = 4$ and $a \neq 4$.
The code thus has the form $ab4d$.
The remaining digits include a 2 (with the restriction that $a \neq 2$), a 7, and either a 1 or a 6.
There are 2 positions in which the 2 can be placed, after which the 7 can be placed in either of the 2 remaining positions, after which there are 2 digits that can be placed in the remaining position.
Therefore, in this case, there are $2 \times 2 \times 2 = 8$ possible codes.

In total, there are $2 + 12 + 8 = 22$ possible codes.

ANSWER: 22

24. We label the two other regions w and z as shown:



If we start with the area of the larger quarter circle (which is equal to $y + w + z$) and then subtract the area of the smaller quarter circle (which is equal to w), we are left $y + z$.
If we then subtract the area of the rectangle (which is equal to $x + z$), we are left with $y - x$.
In other words, $y - x$ is equal to the area of the larger quarter circle minus the area of the smaller quarter circle minus the area of the retangle.
The larger quarter circle has radius 30 and so its area is $\frac{1}{4}\pi \times 30^2 = 225\pi$.
The radius of the smaller quarter circle is half of that of the larger quarter circle, because F is the midpoint of CE.
Thus, the smaller quarter circle has radius 15 and so its area is $\frac{1}{4}\pi \times 15^2 = \frac{225}{4}\pi$.
The width of the rectangle is equal to FC, which is half of CE or 15.

The height of the rectangle is 30, and so its area is $15 \times 30 = 450$.
Therefore, $y - x = 225\pi - \frac{225}{4}\pi - 450 = \frac{900}{4}\pi - \frac{225}{4}\pi - 450 = \frac{675}{4}\pi - 450 \approx 80.1$.
This tells us that $y - x$ is positive (the diagram certainly makes it look positive), which means that $d = y - x$ and the closest integer to d is 80.

ANSWER: 80

25. We write $a = 3^r$, $b = 3^s$ and $c = 3^t$ where each of r, s, t is between 1 and 8, inclusive.
Since $a \le b \le c$, then $r \le s \le t$.
Next, we note that

$$\frac{ab}{c} = \frac{3^r 3^s}{3^t} = 3^{r+s-t} \qquad \frac{ac}{b} = \frac{3^r 3^t}{3^s} = 3^{r+t-s} \qquad \frac{bc}{a} = \frac{3^s 3^t}{3^r} = 3^{s+t-r}$$

Since $t \ge s$, then $r + t - s = r + (t - s) \ge r > 0$ and so $\dfrac{ac}{b}$ is always an integer.
Since $t \ge r$, then $s + t - r = s + (t - r) \ge s > 0$ and so $\dfrac{bc}{a}$ is always an integer.
Since $\dfrac{ab}{c} = 3^{r+s-t}$, then $\dfrac{ab}{c}$ is an integer exactly when $r + s - t \ge 0$ or $t \le r + s$.
This means that we need to count the number of triples (r, s, t) where $r \le s \le t$, each of r, s, t is an integer between 1 and 8, inclusive, and $t \le r + s$.

Suppose that $r = 1$. Then $1 \le s \le t \le 8$ and $t \le s + 1$.
If $s = 1$, t can equal 1 or 2. If $s = 2$, t can equal 2 or 3. This pattern continues so that when $s = 7$, t can equal 7 or 8. When $s = 8$, though, t must equal 8 since $t \le 8$.
In this case, there are $2 \times 7 + 1 = 15$ pairs of values for s and t that work, and so 15 triples (r, s, t).

Suppose that $r = 2$. Then $2 \le s \le t \le 8$ and $t \le s + 2$.
This means that, when $2 \le s \le 6$, t can equal s, $s + 1$ or $s + 2$.
When $s = 7$, t can equal 7 or 8, and when $s = 8$, t must equal 8.
In this case, there are $5 \times 3 + 2 + 1 = 18$ triples.

Suppose that $r = 3$. Then $3 \le s \le t \le 8$ and $t \le s + 3$.
This means that, when $3 \le s \le 5$, t can equal s, $s + 1$, $s + 2$, or $s + 3$.
When $s = 6, 7, 8$, there are 3, 2, 1 values of t, respectively.
In this case, there are $3 \times 4 + 3 + 2 + 1 = 18$ triples.

Suppose that $r = 4$. Then $4 \le s \le t \le 8$ and $t \le s + 4$.
This means that when $s = 4$, there are 5 choices for t.
As in previous cases, when $s = 5, 6, 7, 8$, there are 4, 3, 2, 1 choices for t, respectively.
In this case, there are $5 + 4 + 3 + 2 + 1 = 15$ triples.

Continuing in this way, when $r = 5$, there are $4 + 3 + 2 + 1 = 10$ triples, when $r = 6$, there are $3 + 2 + 1 = 6$ triples, when $r = 7$, there are $2 + 1 = 3$ triples, and when $r = 8$, there is 1 triple.
The total number of triples (r, s, t) is $15 + 18 + 18 + 15 + 10 + 6 + 3 + +1 = 86$.
Since the triples (r, s, t) correspond with the triples (a, b, c), then the number of triples (a, b, c) is $N = 86$.

ANSWER: 86