The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
*cemc.uwaterloo.ca*

# *2024 Cayley Contest*

**(Grade 10)**

**Wednesday, February 28, 2024**
**(in North America and South America)**

**Thursday, February 29, 2024**
**(outside of North America and South America)**

*Solutions*

1. Calculating, $2 \times 0 + 2 \times 4 = 0 + 8 = 8$.

ANSWER: (E)

2. When $x = 3$, we have $-(5x - 6x) = -(-x) = x = 3$.
Alternatively, when $x = 3$, we have $-(5x - 6x) = -(15 - 18) = -(-3) = 3$.

ANSWER: (B)

3. Since $AE = BF$ and $BE = CF$, then $AB = AE + BE = BF + CF = BC$.
Therefore, $\triangle ABC$ is isosceles with $\angle BAC = \angle BCA = 70^\circ$.
Since the sum of the angles in $\triangle ABC$ is $180^\circ$, then

$$\angle ABC = 180^\circ - \angle BAC - \angle BCA = 180^\circ - 70^\circ - 70^\circ = 40^\circ$$

ANSWER: (A)

4. On Friday, the Cayley Comets scored 80% of 90 points.
This is equal to $\frac{80}{100} \times 90 = \frac{8}{10} \times 90 = 8 \times 9 = 72$ points.
Alternatively, since 80% is equivalent to 0.9, then 80% of 90 is equal to $0.8 \times 90 = 72$.

ANSWER: (B)

5. The volume of a prism is equal to the area of its base times its depth.
Here, the prism has identical bases with area 400 cm$^2$ and depth 8 cm, and so its volume is 400 cm$^2$ $\times$ 8 cm = 3200 cm$^3$.

ANSWER: (C)

6. The percentage of cookies that Lloyd ate that were chocolate chip or oatmeal was 33% + 22% which equals 55%.
This leaves 100% − 55% = 45% of the cookies that were gingerbread or sugar.
Since Lloyd ate two times as many gingerbread cookies as sugar cookies, then $\frac{2}{3}$ of the 45%, or 30%, were gingerbread cookies.

ANSWER: (C)

7. Simplifying, $\dfrac{1}{6} + \dfrac{1}{3} = \dfrac{1}{6} + \dfrac{2}{6} = \dfrac{3}{6} = \dfrac{1}{2}$. Thus, $\dfrac{1}{x} = \dfrac{1}{2}$ and so $x = 2$.

ANSWER: (D)

8. Since $4 = 2^2$, then $4^7 = (2^2)^7 = 2^{14} = (2^7)^2$, which means that $4^7$ is a perfect square.
We can check, for example using a calculator, that the square root of each of the other four choices is not an integer, and so each of these four choices cannot be expressed as the square of an integer.

ANSWER: (C)

9. Suppose that the smallest of the five odd integers is $x$.
Since consecutive odd integers differ by 2, the other four odd integers are $x + 2$, $x + 4$, $x + 6$, and $x + 8$.
Therefore, $x + (x + 2) + (x + 4) + (x + 6) + (x + 8) = 125$.
From this, we obtain $5x + 20 = 125$ and so $5x = 105$, which gives $x = 21$.
Thus, the smallest of the five integers is 21. (This means that the five odd integers are 21, 23, 25, 27, 29.)

ANSWER: (C)

10. When two standard six-sided dice are rolled, there are $6 \times 6 = 36$ possibilities for the pair of numbers that are rolled.
 Of these, the pairs $2 \times 6$, $3 \times 4$, $4 \times 3$, and $6 \times 2$ each give 12. (If one of the numbers rolled is 1 or 5, the product cannot be 12.)
 Since there are 4 pairs of possible rolls whose product is 12, the probability that the product is 12 is $\frac{4}{36}$.

 ANSWER: (B)

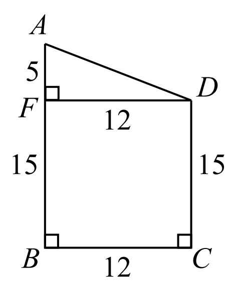
11. Since Arturo has an equal number of \$5 bills, of \$10 bills, and of \$20 bills, then we can divide Arturo's bills into groups, each of which contains one \$5 bill, one \$10 bill, and one \$20 bill.
 The value of the bills in each group is $\$5 + \$10 + \$20 = \$35$.
 Since the total value of Arturo's bills is \$700, then there are $\dfrac{\$700}{\$35} = 20$ groups.
 Thus, Arturo has 20 \$5 bills.

 ANSWER: (D)

12. Since the mass of 2 Exes equals the mass of 29 Wyes, then the mass of $8 \times 2$ Exes equals the mass of $8 \times 29$ Wyes.
 In other words, the mass of 16 Exes equals the mass of 232 Wyes.
 Since the mass of 1 Zed equals the mass of 16 Exes, then the mass of 1 Zed equals the mass of 232 Wyes.

 ANSWER: (C)

13. Draw a perpendicular from $D$ to $F$ on $AB$.
 Since quadrilateral $FDCB$ has right angles at $F$, $C$ and $B$, then it must be a rectangle.
 This means that $FB = DC = 15$ and $FD = BC = 12$.
 Further, $AF = AB - FB = 20 - 15 = 5$.




 Now, $\triangle AFD$ is right-angled at $F$.
 By the Pythagorean Theorem, $AD^2 = AF^2 + FD^2 = 5^2 + 12^2 = 25 + 144 = 169$.
 Since $AD > 0$, then $AD = 13$. (Some might recognize the Pythagorean triple 5-12-13 directly.)
 Thus, the perimeter of $ABCD$ is $20 + 12 + 15 + 13 = 60$.

 ANSWER: (E)

14. Since 10 numbers have an average of 87, their sum is $10 \times 87 = 870$.
 When the numbers 51 and 99 are removed, the sum of the remaining 8 numbers is $870 - 51 - 99$ or 720.
 The average of these 8 numbers is $\dfrac{720}{8} = 90$.

 ANSWER: (A)

15. The sum of the lengths of the horizontal line segments in Figure 2 is $4x$, because the tops of the four small rectangles contribute a total of $2x$ to their combined perimeter and the bottoms of the four small rectangles contribute a total of $2x$ to their combined perimeter.
Similarly, the sum of the lengths of the vertical line segments in Figure 2 is $4y$.
In other words, the sum of the perimeters of the four rectangles in Figure 2 is $4x + 4y$.
Since the sum of the perimeters also equals 24, then $4x + 4y = 24$ and so $x + y = 6$.

ANSWER: (A)

16. Since
$$\sqrt{\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \cdots \times \frac{n-1}{n}} = \frac{1}{8}$$
then squaring both sides, we obtain
$$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \cdots \times \frac{n-1}{n} = \frac{1}{64}$$
Simplifying the left side, we obtain
$$\frac{1 \times 2 \times 3 \times 4 \times \cdots \times (n-1)}{2 \times 3 \times 4 \times 5 \times \cdots \times n} = \frac{1}{64}$$
or
$$\frac{1 \times (2 \times 3 \times 4 \times \cdots \times (n-1))}{(2 \times 3 \times 4 \times \cdots \times (n-1)) \times n} = \frac{1}{64}$$
and so $\dfrac{1}{n} = \dfrac{1}{64}$ which means that $n = 64$.

ANSWER: (B)

17. The integers between 1000 and 9999, inclusive, are all four-digit positive integers of the form $abcd$.
We want each of $a$, $b$, $c$, and $d$ to be even.
There are 4 choices for $a$, namely 2, 4, 6, 8. ($a$ cannot equal 0.)
There are 5 choices for each of $b$, $c$ and $d$, namely 0, 2, 4, 6, 8.
The choice of each digit is independent, and so the total number of such integers is $4 \times 5 \times 5 \times 5$ or 500.

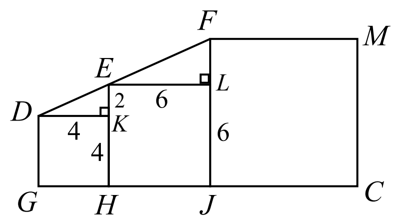
ANSWER: (A)

18. The line with equation $y = 3x + 5$ has slope 3 and $y$-intercept 5.
Since the line has $y$-intercept 5, it passes through $(0, 5)$.
When the line is translated 2 units to the right, its slope does not change and the new line passes through $(2, 5)$.
A line with slope $m$ that passes through the point $(x_1, y_1)$ has equation $y - y_1 = m(x - x_1)$.
Therefore, the line with slope 3 that passes through $(2, 5)$ has equation $y - 5 = 3(x - 2)$ or $y - 5 = 3x - 6$, which gives $y = 3x - 1$.

Alternatively, we could note that when the graph of $y = 3x + 5$ is translated 2 units to the right, the equation of the new graph is $y = 3(x - 2) + 5$ or $y = 3x - 1$.

ANSWER: (B)

19. Since squares $DKHG$, $ELJH$ and $FMCJ$ have their bases along the same line, then $DK$, $EL$ and $FM$ are parallel.
Since $DK$ and $EL$ are parallel, then $\angle EDK = \angle FEL$.
Since $\triangle EKD$ is right-angled at $K$ and $\triangle FLE$ is right-angled at $L$, then $\triangle EKD$ and $\triangle FLE$ are similar.




Since the area of $DKHG$ is 16, then its side length is $\sqrt{16} = 4$.
Since the area of $ELJH$ is 36, then its side length is $\sqrt{36} = 6$.
Since $EH = 6$ and $KH = 4$, then $EK = 2$.
Therefore, $\triangle EKD$ has $EK = 2$ and $DK = 4$; in other words, $EK : DK = 1 : 2$.
Since $\triangle FLE$ is similar to $\triangle EKD$, then $FL : LE = 1 : 2$.
Since $EL = 6$, then $FL = 3$. Since $LJ = 6$ and $FL = 3$, then $FJ = FL + LJ = 9$.
Therefore, the area of square $FMCJ$ is $9^2$ or 81.

Answer: (D)

20. Suppose that the length of the race was $d$ m.
Suppose further that Jiwei finished the first race in $t$ s.
Since Hari finished in $\frac{4}{5}$ of the time that Jiwei took, then Hari finished in $\frac{4}{5}t$ s.
Since speed equals distance divided by time, then Jiwei's average speed was $\frac{d}{t}$ m/s and Hari's average speed was $\frac{d}{4t/5} = \frac{5}{4} \cdot \frac{d}{t}$ m/s.
For Jiwei to finish in the same time as Hari, Jiwei must increase his average speed from $\frac{d}{t}$ m/s to $\frac{5}{4} \cdot \frac{d}{t}$ m/s.
This is an increase of one-quarter over the original speed, or an increase of 25%. Thus, $x = 25$.

Answer: (B)

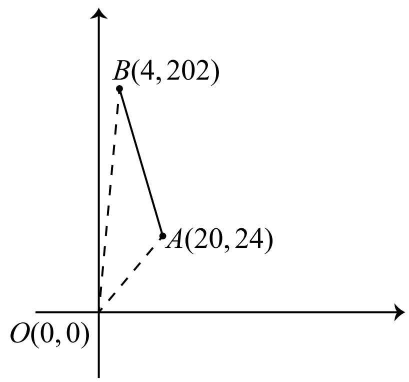
21. Since the second column includes the number 1, then step (ii) was never used on the second column, otherwise each entry would be at least 2.
To generate the 1, 3 and 2 in the second column, we thus need to have used step (i) 1 time on row 1, 3 times on row 2, and 2 times on row 3.
This gives:

| | | |
|---|---|---|
| 1 | 1 | 1 |
| 3 | 3 | 3 |
| 2 | 2 | 2 |

We cannot use step (i) any more times, otherwise the entries in column 2 will increase. Thus, $a = 1 + 3 + 2 = 6$.
To obtain the final grid from this current grid using only step (ii), we must increase each entry in column 1 by 6 (which means using step (ii) 3 times) and increase each entry in column 3 by 4 (which means using step (ii) 2 times). Thus, $b = 3 + 2 = 5$.
Therefore, $a + b = 11$.

ANSWER: 11

22. Let $O$ be the origin, $A$ be the point with coordinates $(20, 24)$, and $B$ be the point with coordinates $(4, 202)$.
The slope of $OA$ is $\frac{24}{20} = 1.2$. The slope of $OB$ is $\frac{202}{4} = 50.5$.




The line with equation $y = mx$ passes through $(0, 0)$.
If $m < 0$, the line with equation $y = mx$ does not pass through the first quadrant.
If $0 \le m < 1.2$, the line with equation $y = mx$ is less steep than $OA$ and so does not intersect line segment $AB$.
If $m > 50.5$, the line with equation $y = mx$ is steeper than $OB$ and so does not intersect line segment $AB$.
For integers $m$ with $2 \le m \le 50$, the line with equation $y = mx$ will intersect the line segment $AB$.
There are 49 such integers $m$.

ANSWER: 49

23. To calculate the shaded area, we add the area of the rectangle and the areas of the four semi-circles, and subtract the area of the larger circle.
Since the rectangle is 6 by 8, its area is $6 \times 8 = 48$.
The two semi-circles of diameter 6 together form a complete circle of diameter 6, or radius 3. The combined area of these semi-circles is $\pi \times 3^2$ or $9\pi$.
The two semi-circles of diameter 8 together form a complete circle of diameter 8, or radius 4. The combined area of these semi-circles is $\pi \times 4^2$ or $16\pi$.
Since the larger circle passes through the four vertices of the rectangle, the diagonal of the rectangle is its diameter. (This is because the diagonal subtends an angle of $90^\circ$ at each of the other vertices and so is a diameter.)
The length of the diagonal is $\sqrt{6^2 + 8^2} = \sqrt{100} = 10$, and so the radius of the larger circle is 5, and so its area is $\pi \times 5^2 = 25\pi$.
Finally, this means that the area of the shaded region is $48 + 9\pi + 16\pi - 25\pi$ which equals 48. The closest integer to 48 is 48.

ANSWER: 48

24. If Rasheeqa walks directly from $A$ to $B$ to $C$, it takes $2 + 3 = 5$ minutes.
We note that Rasheeqa cannot walk from $A$ to $C$ in less than 5 minutes, so $t$ cannot equal 1, 2, 3, or 4.
Rasheeqa can add increments of 3 minutes to her walk by walking around the circular path that begins and ends at $B$.
This means that Rasheeqa's total time, in minutes, can be each of $5, 8, 11, 14, \ldots, 98, 101$.
Rasheeqa can also walk from $A$ to $B$ to $A$ to $B$ to $C$. This takes $2 + 3 + 2 + 3 = 10$ minutes.
Any walk that uses the path from $B$ back to $A$ once must be at least this long.
Rasheeqa can again add increments of 3 minutes to this 10 minute walk by walking around the circular path that begins and ends at $B$.
This means that Rasheeqa's total time, in minutes, can be each of $10, 13, 16, 19, \ldots, 97, 100, 103$.
Rasheeqa can also walk from $A$ to $B$ to $A$ to $B$ to $A$ to $B$ to $C$. This takes $2+3+2+3+2+3 = 15$ minutes.
Any walk that uses the path from $B$ back to $A$ twice must be at least this long.
Rasheeqa can again add increments of 3 minutes to her 15 minute walk by walking around the circular path that begins and ends at $B$.
This means that Rasheeqa's total time, in minutes, can be each of $15, 18, 21, 24, \ldots, 96, 99, 102$.
Examining these lists, we see that $t$ can equal 5, 8, 10, 11, 13, 14, 15, and every integer $t$ with $16 \leq t \leq 100$.
Of the positive integers $t$ with $1 \leq t \leq 100$, we see that $t$ cannot be equal to 1, 2, 3, 4, 6, 7, 9, 12.
Since there are 8 values that $t$ cannot equal, there are $100 - 8 = 92$ possible values of $t$.

ANSWER: 92

25. The patterns that Erin can construct can include 3, 4, 5, or 6 X's.
A pattern cannot include fewer than 3 X's (because 3 X's are required to complete a pattern), and cannot include 7 X's (because the 7th X would be placed next to 6 X's, between 5 X's and 1 X, or between 4 X's and 2 X's, or between 3 X's and 3 X's, each of which would already be a complete pattern).
We consider the following cases which are determined by the number of X's in each pattern.

Case 1: 3 X's
Using O's to represent empty squares, there are 5 ways in which 3 consecutive X's can be placed:

XXXOOOO, OXXXOOO, OOXXXOO, OOOXXXO, OOOOXXX

Case 2: 4 X's
There are 4 ways in which 4 consecutive X's can be placed:

XXXXOOO, OXXXXOO, OOXXXXO, OOOXXXX

(It is possible to have 4 consecutive X's without having stopped after the 3rd X if the X's are placed, for example, in order 1st X, 2nd X, 4th X, 3rd X.)
We can also place 4 X's with a group of 3 X's (we need at least 3 together) and 1 separate X (either after or before). There are 12 ways in which this can be done:

XXXOXOO, XXXOOXO, XXXOOOX, OXXXOXO, OXXXOOX, OOXXXOX, XOXXXOO,
XOOXXXO, XOOOXXX, OXOXXXO, OXOOXXX, OOXOXXX

Case 3: 5 X's
There are 3 ways in which 5 consecutive X's can be placed:

XXXXXOO, OXXXXXO, OOXXXXX

(We can place 5 consecutive X's in the order 1st, 2nd, 4th, 5th, 3rd.)
We can also place 5 X's with a group of 3 X's and a group of 2 X's, or with a group of 4 X's and one indiviudal X, or with a group of 3 X's and two individual X's. There are 15 ways in which this can be done:

XXXOXXO, XXXOOXX, OXXXOXX, XXOXXXO, XXOOXXX, OXXOXXX, XXXXOXO, XXXXOOX, OXXXXOX, XOXXXXO, XOOXXXX, OXOXXXX, XXXOXOX, XOXXXOX, XOXOXXX

Case 4: 6 X's
We cannot place 6 consecutive X's, since there would have to have been 3 consecutive X's before the 6th X was placed.
We can, however, have 6 X's if they are in groups of 5 and 1, or groups of 4 and 2. (We cannot have groups of 3 and 3. Can you see why?)
There are 4 ways in which this can be done:

XXXXXOX, XXXXOXX, XOXXXXX, XXOXXXX

In total, there are $5 + 4 + 12 + 3 + 15 + 4 = 43$ patterns that can be created.

ANSWER: 43