The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2024
Canadian Team Mathematics Contest

April 2024

Solutions

Individual Problems

1. There are 7 days in a week, so $7 \times 3 = 21$ days after May 3 is also Thursday. Since $3 + 21 = 24$, May 24 is a Thursday.
   Thus, May 25 is a Friday, May 26 is a Saturday, May 27 is a Sunday, May 28 is a Monday, and May 29 is a Tuesday.

   ANSWER: Tuesday

2. Observe that $3^2 = 9 < 11 < 16 = 4^2$, so $3 < \sqrt{11} < 4$, which implies that $-4 < -\sqrt{11} < -3$.
   Similarly, $5^2 = 25 < 29 < 36 = 6^2$, so $5 < \sqrt{29} < 6$.
   From the calculations above, an integer n is between $-\sqrt{11}$ and $\sqrt{29}$ exactly when $-3 \leq n \leq 5$.
   There are 9 such integers.

   ANSWER: 9

3. Since A, E, and C lie on a line, $\angle AEC = 180^\circ$, and so $\angle AED + \angle DEC = 180^\circ$.
   Suppose $\angle AED = 7x^\circ$. Then $\angle DEC = 11x^\circ$ because $\angle AED : \angle DEC$ is $7 : 11$. Hence, we conclude that $7x + 11x = 180$.
   Solving this equation for x gives $x = 10$, so $\angle DEC = 11 \times 10^\circ = 110^\circ$.
   The diagonals of a rectangle have equal length and bisect each other, so $DE = CE$.
   Therefore, $\triangle CDE$ is isosceles with $\angle EDC = \angle ECD$.
   If $\angle BDC = y^\circ$, then since $\angle BDC = \angle EDC$, we have $\angle EDC = \angle ECD = y^\circ$.
   The angles in $\triangle CDE$ have a sum of 180°, so $2y + 110 = 180$ or $y = 35$.

   ANSWER: 35°

4. If the common ratio is r, then $x = 80r$, $y = xr = 80r^2$, $z = yr = 80r^3$, and $3125 = zr = 80r^4$.
   Rearranging $3125 = 80r^4$, we get $r^4 = \dfrac{3125}{80} = \dfrac{625}{16} = \dfrac{25^2}{4^2}$.
   Taking square roots, $r^2 = \pm\dfrac{25}{4}$, but r is a real number, so $r^2 \geq 0$, which means $r^2 = \dfrac{25}{4}$.
   Using that $y = 80r^2$ we get $y = 80 \times \dfrac{25}{4} = 500$.
   Although it was not needed to answer the question, there are two possible values of r and they are $r = -\dfrac{5}{2}$ and $r = \dfrac{5}{2}$.

   ANSWER: 500

5. Since x and y are both from -5 to 5 inclusive, the possible values for x^2 and y^2 are $0^2 = 0$, $1^2 = 1$, $2^2 = 4$, $3^2 = 9$, $4^2 = 16$, and $5^2 = 25$.
   In the table below, the possible sums of x^2 and y^2 are summarized. In particular, the integer in the row for y^2 and the column for x^2 is $x^2 + y^2$.

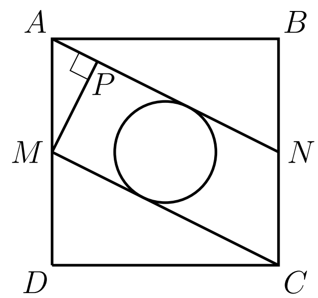
	0	1	4	9	16	25
0	0	1	4	9	16	25
1	1	2	5	10	17	26
4	4	5	8	13	20	29
9	9	10	13	18	25	34
16	16	17	20	25	32	41
25	25	26	29	34	41	50

If a sum in the table above is of the form $x^2 + y^2$ where neither x^2 nor y^2 is zero, then there are four pairs (x, y) leading to that sum.
For example, the integer 20 coming from $x^2 = 4$ and $y^2 = 16$ is achieved by (x, y) equal to each of $(2, 4)$, $(2, -4)$, $(-2, 4)$, and $(-2, -4)$.
If a sum in the table is of the form $x^2 + y^2$ where exactly one of x^2 and y^2 is 0, then there are two pairs (x, y) leading to that sum.
For example, the sum of 16 coming from $x^2 = 0$ and $y^2 = 16$ comes from $(0, 4)$ and $(0, -4)$.
There are 17 sums in the table above that are from 9 through 25.
Of these, 11 come from adding two non-zero squares, and 6 come from adding $0^2 = 0$ to a nonzero square.
From the reasoning above, there are $2 \times 6 + 4 \times 11 = 56$ pairs (x, y) with the given properties.

ANSWER: 56

6. *Solution 1*

Let P be on AN so that MP is perpendicular to AN, as shown below.



Since $ABCD$ is a square, $\angle DAB = 90^\circ$, which means $\angle MAP + \angle NAB = 90^\circ$.
Since $\triangle ABN$ is right-angled at B, we have $\angle NAB + \angle ANB = 90^\circ$.
From $\angle NAB + \angle ANB = 90^\circ = \angle MAP + \angle NAB$, we can deduce that $\angle ANB = \angle MAP$.
As well, $\angle MPA = \angle ABN = 90^\circ$, so we conclude that $\triangle MPA$ and $\triangle ABN$ are similar.

Since $ABCD$ is a square, $BC = AB = 2$, and since N is the midpoint of BC, we have $BN = 1$.
Applying the Pythagorean theorem to $\triangle ABN$ gives $AN = \sqrt{AB^2 + BN^2} = \sqrt{2^2 + 1^2} = \sqrt{5}$.
By the similarity of $\triangle MPA$ and $\triangle ABN$, we have that $\dfrac{MP}{AM} = \dfrac{AB}{AN} = \dfrac{2}{\sqrt{5}}$ or $MP = \dfrac{2AM}{\sqrt{5}}$.
M is the midpoint of AD, so $AM = 1$, hence $MP = \dfrac{2}{\sqrt{5}}$.
The lines AN and MC are parallel, so the diameter of the circle is the perpendicular distance between these two lines, which is the length of MP.
Therefore, the radius of the circle is $\dfrac{MP}{2} = \dfrac{1}{\sqrt{5}}$ and its area is $\pi\left(\dfrac{1}{\sqrt{5}}\right)^2 = \dfrac{\pi}{5}$.

Solution 2

We will coordinatize to get $A(0, 2)$, $B(2, 2)$, $C(2, 0)$, and $D(0, 0)$, from which it follows that M is at $(0, 1)$ since it is the midpoint of AD. Similarly, N is at $(2, 1)$.
Line segment AN has slope $\dfrac{1-2}{2-0} = -\dfrac{1}{2}$ and line segment MC has slope $\dfrac{0-1}{2-0} = -\dfrac{1}{2}$.
Therefore, AN and MC are parallel. By symmetry and because these lines are parallel, the centre of the circle must be on the line of slope $-\dfrac{1}{2}$ that is half way between AN and MC.
This line must pass through the midpoint of AM, which is $\left(0, \dfrac{3}{2}\right)$ so the centre of the circle

lies somewhere on the line with equation $y = -\frac{1}{2}x + \frac{3}{2}$.
Therefore, for some a, the centre of the circle is at $\left(a, \frac{3-a}{2}\right)$.
The circle has equation $(x-a)^2 + \left(y - \frac{3-a}{2}\right)^2 = r^2$ where r is the radius of the circle.
The line through $M(0,1)$ and $C(2,0)$ has equation $y = -\frac{1}{2}x + 1$.
Line segment MC is tangent to the circle, so it intersects the circle exactly once.
Substituting $y = -\frac{1}{2}x + 1$ into the equation of the circle gives the following equivalent equations:

$$\begin{aligned}
(x-a)^2 + \left(-\frac{1}{2}x + 1 - \frac{3-a}{2}\right)^2 &= r^2 \\
4(x-a)^2 + (-x+2-3+a)^2 &= 4r^2 \\
4(x-a)^2 + (-x+a-1)^2 &= 4r^2 \\
4x^2 - 8ax + 4a^2 + x^2 + a^2 + 1 - 2ax + 2x - 2a &= 4r^2 \\
5x^2 + (2-10a)x + 5a^2 - 2a + 1 - 4r^2 &= 0
\end{aligned}$$

The x-values that satisfy this equation represent the points of intersection of the circle with MC. There is only one such point, so the quadratic above must have only one root.
A quadratic with only one root has a discriminant equal to 0, so we get the equivalent equations

$$\begin{aligned}
(2-10a)^2 - 4(5)(5a^2 - 2a + 1 - 4r^2) &= 0 \\
4 - 40a + 100a^2 - 100a^2 + 40a - 20 + 80r^2 &= 0 \\
80r^2 &= 16
\end{aligned}$$

Solving $80r^2 = 16$ for r^2 gives $r^2 = \frac{1}{5}$, so the area of the circle is $\pi r^2 = \frac{\pi}{5}$.

Answer: $\frac{\pi}{5}$

7. The amount of money, in cents, that Rolo has is $5x + 10y + 25z$.
The amount of money, in cents, that Pat has is $5y + 10z + 25x$.
The amount of money, in cents, that Sarki has is $5z + 10x + 25y$.
Therefore, the total amount of money that they have together is

$$(5x + 10y + 25z) + (5y + 10z + 25x) + (5z + 10x + 25y) = 40x + 40y + 40z$$

They have 6480 cents in total, which leads to the equation $40(x + y + z) = 6480$.
Dividing both sides by 40, we get $x + y + z = 162$.
Since Pat has y nickels, z dimes, and x quarters, she has $x + y + z = 162$ coins in total.

Answer: 162

8. There are five odd digits: 1, 3, 5, 7, and 9.
We will count how many integers less than 1000 with only odd digits have 1 as a units digit.
The integer 1 is one of these integers, and the integers 11, 31, 51, 71, and 91 are the two-digit integers with 1 as their units digit.
There are $5 \times 5 = 25$ three-digit integers with only odd digits and 1 as their units digit. This is because there are 5 choices for the hundreds digit and 5 choices for the tens digit, and these choices are independent.

Thus, $25 + 5 + 1 = 31$ integers less than 1000 with only odd digits have a units digit of 1.
By nearly identical reasoning, for each possible odd digit, there are 31 integers less than 1000 with only odd digits and that units digit.
When we add all integers less than 1000 that have only odd digits, the units digits contribute

$$31(1 + 3 + 5 + 7 + 9) = 31(25) = 775$$

Doing similar analysis on the tens digits, we find that for any given odd digit, there are exactly 5 two-digit integers with only odd digits and that tens digit, and there are $5 \times 5 = 25$ three-digit integers with that given tens digit. There are no one-digit integers with an odd tens digit.
Therefore, for each odd digit, there are $5 + 25 = 30$ integers less than 1000 with only odd digits and that given tens digit.
When we add all integers less than 1000 that have only odd digits, the tens digits contribute

$$30(10 + 30 + 50 + 70 + 90) = 7500$$

Similarly, there are $5 \times 5 = 25$ three-digit integers with only odd digits for each possible odd hundreds digit. (An integer with an odd hundreds digit has at least three digits.)
When we add all integers less than 1000 that have only odd digits, the hundreds digits contribute

$$25(100 + 300 + 500 + 700 + 900) = 62\,500$$

Therefore, the sum of all integers less than 1000 that have only odd digits is

$$62\,500 + 7500 + 775 = 70\,775$$

Answer: 70775

9. Suppose the radius of the sphere is r and let E be any point on the circumference of the circular cross section in the hemisphere containing A.
Then $\triangle EAO$ has $\angle EAO = 90^\circ$ and $EO = r$. As well, since $AC = r$, so $AO = \dfrac{AC}{3} = \dfrac{1}{3}r$.
By the Pythagorean theorem, $AE^2 + AO^2 = EO^2$ so $AE^2 = r^2 - \left(\dfrac{1}{3}r\right)^2 = \dfrac{8}{9}r^2$.
The cone with A in its base has base radius AE and height AO, so its volume is

$$\frac{1}{3}\pi(AE)^2(AO) = \frac{1}{3}\pi\left(\frac{8}{9}r^2\right)\frac{r}{3} = \frac{8\pi}{81}r^3$$

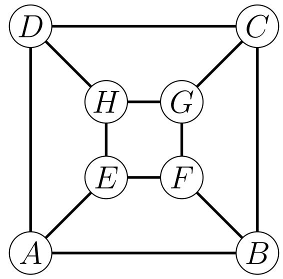
Let F be any point on the circumference of the circular cross section containing B.
Similar to the situation above, we have that $\triangle FBO$ has $\angle FBO = 90^\circ$, $FO = r$, and $BO = \dfrac{2}{3}r$.
By the Pythagorean theorem, we get $BF^2 = r^2 - \left(\dfrac{2}{3}r\right)^2 = \dfrac{5}{9}r^2$.
The cone with B in its base has base radius BF and height BO, so its volume is

$$\frac{1}{3}\pi BF^2 BO = \frac{1}{3}\pi\left(\frac{5}{9}r^2\right)\frac{2}{3}r = \frac{10\pi}{81}r^3$$

Therefore, the ratio we seek is $\left(\dfrac{8\pi}{81}r^3\right) : \left(\dfrac{10\pi}{81}r^3\right)$ which simplifies to $4 : 5$.

Answer: $4 : 5$

10. Label the vertices of the cube by A, B, C, D, E, F, G, and H. The vertices of the cube and the edges connected them can be represented by the following diagram:



Throughout this solution, we will refer to ants by the vertex from which they originate. For example, Ant A is the ant that is originally at vertex A.
Each ant has three choices of which vertex it can walk to. These choices are summarized in the table below.

Ant	Choices
A	B, D, E
B	A, C, F
C	B, D, G
D	A, C, H
E	A, F, H
F	B, E, G
G	C, F, H
H	D, E, G

Since each ant has three choices and their choices are independent, there are 3^8 possible ways that the 8 ants can choose vertices. We will count the number of ways that avoid a collision and get the answer by dividing this total by 3^8.

By symmetry, if we assume that Ant A chooses B, we will count exactly one third of the possibilities. In this case, Ant B cannot choose A without colliding with Ant A, so Ant B must choose C or F. By symmetry, there are an equal number of choices for each case.

Therefore, we will assume that Ant A chooses B and that Ant B chooses C. This will count exactly one sixth of the choices that avoid a collision.

Ant C cannot choose B, so it must choose either D or G. For now, suppose Ant C chooses G. The choices for Ant F are B, E, and G, but B and G were chosen by other ants, so Ant F must choose E.
The choices for Ant H are D, E, and G, but E and G were chosen by other ants, so Ant H must choose D.
To summarize the choices so far, the table below is the same as the table above, but with choices crossed out if either the Ant did not choose that vertex or cannot choose that vertex because another ant has already chosen it.

Ant	Choices
A	$B, \cancel{D}, \cancel{E}$
B	$\cancel{A}, C, \cancel{F}$
C	$\cancel{B}, \cancel{D}, G$
D	$A, \cancel{C}, H$
E	A, F, H
F	$\cancel{B}, E, \cancel{G}$
G	$\cancel{C}, F, H$
H	$D, \cancel{E}, \cancel{G}$

The remaining choices for Ant D are A and H, but Ant D cannot choose H without colliding with Ant H on an edge.
Therefore, Ant D must choose A, and we can update the table as follows, eliminating A as a choice for Ant E.

Ant	Choices
A	$B, \cancel{D}, \cancel{E}$
B	$\cancel{A}, C, \cancel{F}$
C	$\cancel{B}, \cancel{D}, G$
D	$A, \cancel{C}, \cancel{H}$
E	$\cancel{A}, F, H$
F	$\cancel{B}, E, \cancel{G}$
G	$\cancel{C}, F, H$
H	$D, \cancel{E}, \cancel{G}$

Since Ant F chose E, Ant E cannot choose F since they would collide on an edge. Therefore, the remaining choices are that Ant E must choose H and Ant G must choose F. One can verify that the choices have the property that every vertex is chosen by exactly one ant, and no two ants choose each other's original vertex, so there is no collision.
We have now deduced that if Ant C chooses G, then there is only one way for the rest of the ants to choose vertices so that no collision happens.

From now on, we will assume that Ant C chooses D. With Ant A choosing B, and B choosing C, and Ant C choosing D, the choices for the ants are restricted as shown in the table below. Choices have been eliminated if the vertex was chosen by another ant, or if choosing that vertex would cause two ants to "swap" vertices and collide on an edge.

Ant	Choices
A	$B, \cancel{D}, \cancel{E}$
B	$\cancel{A}, C, \cancel{F}$
C	$\cancel{B}, D, \cancel{G}$
D	$A, \cancel{C}, H$
E	A, F, H
F	$\cancel{B}, E, G$
G	$\cancel{C}, F, H$
H	$\cancel{D}, E, G$

We will now consider the two remaining possibilities for the choice of Ant D.
If Ant D chooses H, then the same sort of reasoning that was used earlier leads to the table below, showing that there is only one possibility if Ant D chooses H.

Ant	Choices
A	B, $\cancel{D}$, $\cancel{E}$
B	$\cancel{A}$, C, $\cancel{F}$
C	$\cancel{B}$, D, $\cancel{G}$
D	$\cancel{A}$, $\cancel{C}$, H
E	A, $\cancel{F}$, $\cancel{H}$
F	$\cancel{B}$, E, $\cancel{G}$
G	$\cancel{C}$, F, $\cancel{H}$
H	$\cancel{D}$, $\cancel{E}$, G

Now we assume that Ant D chooses A and reduce the possibilities as follows.

Ant	Choices
A	B, $\cancel{D}$, $\cancel{E}$
B	$\cancel{A}$, C, $\cancel{F}$
C	$\cancel{B}$, D, $\cancel{G}$
D	A, $\cancel{C}$, $\cancel{H}$
E	$\cancel{A}$, F, H
F	$\cancel{B}$, E, G
G	$\cancel{C}$, F, H
H	$\cancel{D}$, E, G

There are two choices for E, and each of them leads to a unique way of the rest of the ants choosing vertices. If Ant E chooses F, we get the table below on the left, and if Ant E chooses H, we get the table below and on the right.

Ant	Choices
A	B, $\cancel{D}$, $\cancel{E}$
B	$\cancel{A}$, C, $\cancel{F}$
C	$\cancel{B}$, D, $\cancel{G}$
D	A, $\cancel{C}$, $\cancel{H}$
E	$\cancel{A}$, F, $\cancel{H}$
F	$\cancel{B}$, $\cancel{E}$, G
G	$\cancel{C}$, $\cancel{F}$, H
H	$\cancel{D}$, E, $\cancel{G}$

Ant	Choices
A	B, $\cancel{D}$, $\cancel{E}$
B	$\cancel{A}$, C, $\cancel{F}$
C	$\cancel{B}$, D, $\cancel{G}$
D	A, $\cancel{C}$, $\cancel{H}$
E	$\cancel{A}$, $\cancel{F}$, H
F	$\cancel{B}$, E, $\cancel{G}$
G	$\cancel{C}$, F, $\cancel{H}$
H	$\cancel{D}$, $\cancel{E}$, G

To summarize, if we assume that A chooses B and B chooses C, then we will count one sixth of the possibilities.
Considering cases, if Ant C chooses G, then there is exactly one choice for the rest of the ants that will avoid collisions.
If Ant C chooses D, then there is one possibility if Ant D chooses H, and two possibilities if Ant D chooses A.
This means $1 + 1 + 2 = 4$ is one sixth of the total number of possibilities, so the answer is

$$\frac{6 \times 4}{3^8} = \frac{8}{3^7} = \frac{8}{2187}$$

Answer: $\frac{8}{3^7}$ $\left(\frac{8}{2187}\right)$

Team Problems

1. Since an hour consists of $3 \times 20 = 60$ minutes, the cyclist will travel $3 \times 8 = 24$ km in 1 hour. Therefore, the cyclist will travel $2 \times 24 = 48$ km in two hours.

   ANSWER: 48 km

2. Squaring both sides, we get $x + 5 = 5^2 = 25$, which can be rearranged to get $x = 20$.

   ANSWER: 20

3. The units digit of $2A7 \times 3$ is the same as the units digit of $7 \times 3 = 21$, so $B = 1$.
   Therefore, $2A7 \times 3 = 711$, so $2A7 = \frac{711}{3} = 237$, so $A = 3$.
   The sum of A and B is $A + B = 3 + 1 = 4$.

   ANSWER: 4

4. In order to save money, k individual rides needs to cost more than \$90.
   This means the answer to the question is the smallest integer k for which $k \times (\$3.25) > \$90$.
   If $3.25k > 90$, then $k > \frac{90}{3.25} = \frac{9000}{325} = \frac{360}{13} = 27 + \frac{9}{13}$.
   The smallest integer k such that $k > 27 + \frac{9}{13}$ is $k = 28$.

   ANSWER: 28

5. In this solution, we will refer to the squares by their label.
   From the diagram, we can see that **E** is the only completely visible square, so it must have been the last to be placed on the table.
   Some of **G** is covering some of **F**, so **G** must have been placed later than **F**.
   Some of **F** is covering some of **D**, so **F** must have been placed later than D.
   By similar reasoning, **D** must have been placed later than **B**, which must have been placed later than both **A** and **C**.
   Finally, **A** is partially covering **C**, so **A** was placed after **C**.
   Putting this all together, the papers were placed in the following order, given by their labels: **C**, **A**, **B**, **D**, **F**, **G**, **E**.

   ANSWER: **CABDFGE**

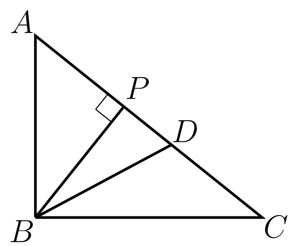
6. The equation of the first line can be rearranged to get $6y = 2x + 42$ or $y = \frac{1}{3}x + 7$, so the slope of the first line is $\frac{1}{3}$.
   If $k = 0$, then the other line is vertical and is not perpendicular to a line with slope $\frac{1}{3}$.
   Therefore, the equation of the second line can be rearranged to $y = -\frac{15}{k}x - \frac{d}{k}$.
   The slopes of perpendicular lines have a product of -1 (unless they are vertical and horizontal), so $\left(\frac{1}{3}\right) \times \left(\frac{-15}{k}\right) = -1$ or $-\frac{5}{k} = -1$, so $k = 5$.
   The lines intersect on the y-axis, which means the two lines intersect when $x = 0$.
   Substituting $k = 5$ and $x = 0$ into the two equations, we get $-6y + 42 = 0$ and $5y + d = 0$.
   The equation $-6y + 42 = 0$ can be solved to get $y = 7$. Substituting into $5y + d = 0$ gives $5(7) + d = 0$ or $d = -35$.

   ANSWER: -35

7. Suppose $x = AB$, meaning that the tens digit of x is A and the units digit of x is B so that $x = 10A + B$.
Then $y = BA$ and we have $18 = y - x = 10B + A - (10A + B) = 9B - 9A$.
Dividing both sides by 9 gives $2 = B - A$.
We cannot have $A = 0$ because x would not be a two-digit integer. Therefore, the smallest that A can be is $A = 1$, which means $B = A + 2 = 3$.
The integer B is a digit, so $B \leq 9$, which implies $A + 2 \leq 9$ or $A \leq 7$.
Therefore, A can take on the values 1, 2, 3, 4, 5, 6, and 7, for a total of 7 possible values of A, and hence, 7 possible values of x.

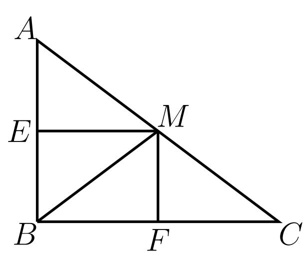
ANSWER: 7

8. Let P be on AC so that BP is perpendicular to AC.



Then $\triangle ADB$ and $\triangle CDB$ have the same altitude, BP.
Since they have equal areas and altitudes, they must also have equal bases, so $AD = CD$. In other words, D is the midpoint of AC.
By a fact about right-triangles that we will prove below, the midpoint of the hypotenuse is equidistant from the three vertices of the triangle, so we conclude that $AD = CD = BD$, but $AD = \frac{AC}{2} = \frac{5}{2}$, so $BD = \frac{5}{2}$.

We now prove that if $\triangle ABC$ has a right angle at B, then the midpoint of AC is equidistant from A, B, and C.
Suppose M is the midpoint of AC and let E and F be on AB and BC, respectively, so that AB is perpendicular to EM and BC is perpendicular to FM.



Since $BEMF$ has three right angles, it must have four right angles and be a rectangle, which implies $BE = FM$.
Since EM and FC are parallel, we get that $\angle EMA = \angle FCM$.
Since EA and FM are parallel, we get that $\angle EAM = \angle FMC$.
By construction, M is the midpoint of AC, so we also have that $AM = MC$.
Therefore, $\triangle AEM$ and $\triangle MFC$ are congruent by angle-side-angle congruence.
From this congruence, we get that $EM = FC$, but $EM = BF$, so we have $BF = FC$.
Consider $\triangle MFB$ and $\triangle MFC$.
These triangles share side MF and we have just shown that $BF = FC$.

We also have that MF is perpendicular to BC, so $\angle BFM = \angle CFM$.
Therefore, $\triangle BFM$ and $\triangle CFM$ are congruent by side-angle-side congruence.
From this congruence, we conclude that $BM = CM$, but since M is the midpoint of AC, we also have $CM = AM$, so $AM = BM = CM$.

Answer: $\frac{5}{2}$

9. Let m be the number rolled on the 8-sided die and n be the number rolled on the 9-sided die. We will count ordered pairs (m, n) with the property that mn is a multiple of 6 and divide this total by $8 \times 9 = 72$, the number of possible rolls, to get the probability.
Suppose $m = 1$. Then for mn to be a multiple of 6, we must have that n is a multiple of 6. The only multiple of 6 between 1 and 9 inclusive is 6, so if $m = 1$, we must have $n = 6$. Therefore, we get the ordered pair $(1, 6)$.
If $m = 5$ or $m = 7$, then we also must have $n = 6$, so we get the pairs $(5, 6)$ and $(7, 6)$.
If $m = 2$, then n must be a multiple of 3, so $n = 3$, $n = 6$, or $n = 9$. We get 3 pairs in this case: $(2, 3)$, $(2, 6)$, and $(2, 9)$.
By similar reasoning, there are three pairs for each of $m = 4$ and $m = 8$.
If $m = 3$, then n must be even, so $n = 2$, $n = 4$, $n = 6$, or $n = 8$, for a total of 4 possibilities.
If $m = 6$, then mn is always a multiple of 6, so there are 9 possibilities.
The table below summarizes the work above.

m	# of n for which mn is a multiple of 6
1	1
2	3
3	4
4	3
5	1
6	9
7	1
8	3

The sum of the numbers in the right column of the table is 25, so the probability is $\dfrac{25}{72}$.

Answer: $\frac{25}{72}$

10. The divisors of 120 are

$$\pm1, \pm2, \pm3, \pm4, \pm5, \pm6, \pm8, \pm10, \pm12, \pm15, \pm20, \pm24, \pm30, \pm40, \pm60, \pm120$$

of which there are 32.
If a is any of the 16 positive divisors of 120, then $b = \dfrac{120}{a}$ is also positive and $a + b > 0 > -23$. Therefore, for each of the 16 positive divisors a of 120, there is exactly one pair that satisfies the given conditions.
By similar reasoning, if a is negative, then so is b. Therefore, if either a or b is any of -24, -30, -40, -60, or -120, then $a + b$ will be less than -23.
As well, in any divisor pair of 120, one of the two divisors is at least 12 in absolute value. Therefore, one of a and b must be -12, -15, or -20.
If $a = -12$, then $b = -10$ and $a + b = -22 > -23$, so we get the pair $(a, b) = (-12, -10)$.
If $a = -15$, then $b = -8$, so $a + b = -23$ which is not greater than -23.
If $a = -20$, then $b = -6$, so $a + b = -26$ which is not greater than -23.

Similarly, if $b = -12$ then $a = -10$ and we get the pair $(a, b) = (-10, -12)$, and if $b = -15$ or $b = -20$, then $a + b$ is not greater than -23.
From earlier, there are 16 pairs where a and b are both positive. We found two additional pairs when a and b are both negative, and there are no other pairs, so the answer is $16 + 2 = 18$.

ANSWER: 18

11. A parabola with equation $y = ax^2 + bx + c$ with $a < 0$ has its maximum y-value at $x = -\dfrac{b}{2a}$.
For the given parabola, the maximum y value occurs at $x = -\dfrac{4k}{-4} = k$.
The maximum y-value is given to be 48, so we must have

$$48 = -2(k)^2 + 4k(k) - 10k = -2k^2 + 4k^2 - 10k = 2k^2 - 10k$$

which can be rearranged to get $2k^2 - 10k - 48 = 0$.
Dividing through by 2 gives $k^2 - 5k - 24 = 0$, which can be factored to get $(k - 8)(k + 3) = 0$.
Therefore, the possible values of k are $k = 8$ and $k = -3$.
When $k = 8$, the parabola has equation $y = -2x^2 + 32x - 80$. This parabola has a maximum y value at $x = 8$ and a maximum y value of $-2(8)^2 + 32(8) - 80 = 48$.
When $k = -3$, the parabola has equation $y = -2x^2 - 12x + 30$. This parabola has a maximum y value at $x = -3$ and a maximum y value of $-2(-3)^2 - 12(-3) + 30 = 48$.
Therefore, the only possible values of k are $k = 8$ and $k = -3$, which have a sum of 5.

ANSWER: 5

12. We want to find $A + B + C + D$, so we first set $x = A + B + C + D$.
The given information leads to the following three equations:

$$\begin{aligned} \frac{5}{2} \times \frac{x}{4} &= A \\ \frac{4}{5} \times \frac{x}{4} &= B \\ 140 &= C + D \end{aligned}$$

Adding these equations gives

$$\frac{5}{8}x + \frac{1}{5}x + 140 = A + B + C + D$$

Using $x = A + B + C + D$, we get $\frac{5}{8}x + \frac{1}{5}x + 140 = x$.
Rearranging this equation, we get $x\left(1 - \frac{5}{8} - \frac{1}{5}\right) = 140$, or $\frac{7}{40}x = 140$.
Multiplying both sides by $\frac{40}{7}$ gives $x = 800$.

ANSWER: 800

13. Cubing both sides of the given equation, we get $\dfrac{3^8 + 3^n}{3^2 + 3^n} = 3^3$.
Clearing the denominator gives $3^8 + 3^n = 3^5 + 3^3 3^n$, and rearranging gives $3^n(3^3 - 1) = 3^8 - 3^5$.
Factoring 3^5 out of the right side of the equation above gives $3^n(3^3 - 1) = 3^5(3^3 - 1)$. After dividing by $3^3 - 1$, we get $3^n = 3^5$, and so $n = 5$.

ANSWER: 5

14. We begin by computing a few terms of the sequence to look for a pattern.
When $n = 3$, $n - 1 = 2$ and $t_2 = 4$, so t_{n-1} is even. Therefore, $t_3 = \frac{1}{2}t_2 + t_1 = 2 + 3 = 5$.
When $n = 4$, $t_{n-1} = 5$ is odd, so $t_4 = t_3 - t_2 = 5 - 4 = 1$.
When $n = 5$, $t_{n-1} = 1$ is odd, so $t_5 = t_4 - t_3 = 1 - 5 = -4$.
When $n = 6$, $t_{n-1} = -4$ is even, so $t_6 = \frac{1}{2}(-4) + 1 = -1$.
When $n = 7$, $t_{n-1} = -1$ is odd, so $t_7 = -1 - (-4) = 3$.
When $n = 8$, $t_{n-1} = 3$ is odd, so $t_8 = 3 - (-1) = 4$.
Since each term in the sequence depends only on the previous two terms, the fact that we have a 4 following a 3 again means that the sequence is periodic. Specifically, the sequence is given by the six terms 3, 4, 5, 1, -4, -1 repeating in that order.
Since $2024 = 6(337) + 2$, the first 2024 terms in the sequence are 337 copies of the terms 3, 4, 5, 1, -4, and -1, followed by an additional 3 and an additional 4.
Therefore, the sum of the first 2024 terms is

$$337(3 + 4 + 5 + 1 - 4 - 1) + 3 + 4 = 337(8) + 7 = 2703$$

Answer: 2703

15. For now, consider only the first five letters.
There are two possibilities for the second letter. There are also two possibilities for the fourth letter since C and E are the only two letters that can be followed by D.
Therefore, if a sequence of five letters starts with A and ends with D, then it must be configured in one of the following four ways:

$$\text{A, B, \_\_, C, D} \qquad \text{A, B, \_\_, E, D} \qquad \text{A, C, \_\_, C, D} \qquad \text{A, C, \_\_, E, D}$$

The first configuration above is impossible because there is no letter with the property that it can both follow B and be followed by C. The third and fourth configurations are impossible by similar reasoning.
Looking at the second configuration, the only letter that can follow B and be followed by E is C, and so we conclude that A, B, C, E, D is the only sequence of five letters starting with A and ending with D.
Therefore, the first five letters must be A, B, C, E, and D in that order.
Using similar reasoning, since the fifth letter is D and the ninth letter is A, there are four possible configurations of the fifth through ninth letters:

$$\text{D, A, \_\_, D, A} \qquad \text{D, A, \_\_, E, A} \qquad \text{D, B, \_\_, D, A} \qquad \text{D, B, \_\_, E, A}$$

We will examine these four configurations separately.
For the first configuration, C is the only letter that can follow A and be followed by D, so one possible sequence is D, A, C, D, A.
For the second configuration, both B and C can follow A and be followed by E, so we get D, A, B, E, A and D, A, C, E, A as possible sequences.
For the third configuration, both C and E can follow B and be followed by D, so we get D, B, C, D, A and D, B, E, D, A as possible sequences.
For the fourth configuration, C is the only letter that can follow B and be followed by E , so the only possible sequence is D, B, C, E, A in this case.
There are six possibilities for the last five letters and only one for the first five, so there are six possible sequence in total.

Answer: 6

16. If Ferd removes Tile G, then the board will be configured as shown below, and it can be checked that every available move with this configuration will cause his opponent to lose. Therefore, Ferd will win if he removes Tile G.

A	B		D
E	F		H
	J	K	L

We will now show that every other move will give his opponent the ability to guarantee that they win.

Currently, Tiles B, D, G, E, and J share an edge with an empty square. If Ferd removes a tile that shares an edge with any of these six tiles, then he will lose the game.
Therefore, if Ferd removes Tile A, Tile F, Tile H, or Tile K, he will lose.
Consider the three configurations below. In the first, Tiles B and L have been removed, in the second, Tiles E and L have been removed, and in the third, Tiles D and J have been removed.

A			D
E	F	G	H
	J	K	

A	B		D
	F	G	H
	J	K	

A	B		
E	F	G	H
		K	L

Notice that in each of the three configurations above, no remaining tile shares an edge with with at least two empty squares.
As well, it can be verified that removing any tile from any of these configurations will cause at least one tile to share an edge with at least two empty squares.
Therefore, if a player is faced with the board in any of the three configurations above, they will lose the game.
If Ferd removes any of Tiles B, D, E, J, or L, his opponent can guarantee a win.
To see why, consider the possibility that Ferd removes Tile B. Then his opponent can remove Tile L to leave him in with the first configuration above.
His opponent will have a similar move if Ferd removes any of Tiles D, E, J, or L, according to one of the three configurations.
We now have that Ferd will lose (provided his opponent makes the correct next move) if he removes Tile A, Tile B, Tile D, Tile E, Tile F, Tile H, Tile J, Tile K, or Tile L. This means he must remove Tile G to have any chance of winning.

ANSWER: Tile G

17. Using exponent rules, $4^{\sin^2 x} = 2^{2\sin^2 x}$ and so $\left(4^{\sin^2 x}\right)\left(2^{\cos^2 x}\right) = 2^{2\sin^2 x + \cos^2 x}$.
As well, $2\sqrt{2} = 2^{\frac{3}{2}}$, so we have $2^{2\sin^2 x+\cos^2 x} = 2^{\frac{3}{2}}$ which implies $2\sin^2 x + \cos^2 x = \dfrac{3}{2}$.

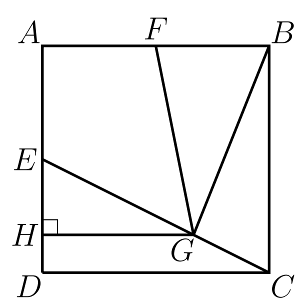
Using that $\sin^2 x + \cos^2 x = 1$, we have

$$\begin{aligned} 2\sin^2 x + \cos^2 x &= \frac{3}{2} \\ \sin^2 x + (\sin^2 x + \cos^2 x) &= \frac{3}{2} \\ \sin^2 x + 1 &= \frac{3}{2} \end{aligned}$$

from which it follows that $\sin^2 x = \frac{1}{2}$, and so $\sin x = \frac{1}{\sqrt{2}}$ or $\sin x = -\frac{1}{\sqrt{2}}$.
The values of x with $0^\circ \le x \le 360^\circ$ for which $\sin x = \pm\frac{1}{\sqrt{2}}$ are $x = 45^\circ$, $x = 135^\circ$, $x = 225^\circ$, and $x = 315^\circ$.
Every value of x with $\sin x = \pm\frac{1}{\sqrt{2}}$ is equal to one of these four values plus a multiple of 360°.
Since $315^\circ + 5 \times 360^\circ = 2115^\circ$ is larger than 2024° but $315^\circ + 4 \times 360^\circ = 1755^\circ$ is less than 2024°, we can add $k \times 360^\circ$ to the four angles for $k = 0$, $k = 1$, $k = 2$, $k = 3$, and $k = 4$ and get a value of x in the range $0^\circ \le x \le 2024^\circ$.
This gives $4 \times 5 = 20$ x-values.
We also get $x = 45^\circ + 5 \times 360^\circ = 1845^\circ$ and $x = 135^\circ + 5 \times 360^\circ = 1935^\circ$ in the desired range, but $225^\circ + 5 \times 360^\circ = 2025^\circ$ is too large, so $x = 1935^\circ$ is the largest possible value.
This gives a total of $20 + 2 = 22$ possible values of x in the given range.

ANSWER: 22

18. In the diagram below, H is on AD so that GH is perpendicular to AD.



Since F is the midpoint of AB, we have $FB = \frac{14}{2} = 7$.
The height of $\triangle FGB$ with base FB is equal in length to AH.
Since the ratio of the area of $\triangle FGB$ to the area of $ABCD$ is $5 : 28$, we have

$$\frac{5}{28} = \frac{\frac{1}{2}(FB)(AH)}{AB^2} = \frac{\frac{1}{2}(7)(AH)}{14^2} = \frac{AH}{56}$$

From $\frac{5}{28} = \frac{AH}{56}$, we get $AH = 10$.
Since $AE = \frac{AD}{2} = \frac{14}{2}$, we get $EH = AH - AE = 10 - 7 = 3$.
The line segments HG and DC are parallel, so $\angle HGE = \angle DCE$.
As well, $\angle EHG = \angle EDC = 90^\circ$, so we get that $\triangle EHG$ and $\triangle EDC$ are similar by angle-angle similarity.

Thus, we have $\dfrac{EH}{ED} = \dfrac{EG}{EC}$, but $EH = 3$ and $ED = \dfrac{14}{2} = 7$, so $\dfrac{EG}{EC} = \dfrac{3}{7}$ or $EC = \dfrac{7}{3}EG$.

If we let $EG = x$, then we have $EC = \dfrac{7}{3}x$, and so $GC = EC - EG = \dfrac{7}{3}x - x = \dfrac{4}{3}x$.

Therefore, $EG : GC$ is $x : \dfrac{4}{3}x$, which can be simplified to $3 : 4$.

Answer: $3 : 4$

19. *Solution 1*

Let $u = x - \dfrac{1}{x}$ and observe that

$$\begin{aligned} u^3 + 3u &= \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right) \\ &= x^3 - 3x + \frac{3}{x} - \frac{1}{x^3} + 3x - \frac{3}{x} \\ &= x^3 - \frac{1}{x^3} \\ &= f(u) \end{aligned}$$

So we have that $f(u) = u^3 + 3u$, provided u is a real number of the form $u = x - \dfrac{1}{x}$.

Setting $1 = x - \dfrac{1}{x}$ and rearranging gives $x^2 - x - 1 = 0$, which has two real roots since the discriminant of the quadratic is $(-1)^2 - 4(-1) = 5 > 0$.

Therefore, $u = 1$ is of the given form, so $f(1) = 1^3 + 3(1) = 4$.

Solution 2

We will first find a value of x for which $1 = x - \dfrac{1}{x}$.

Rearranging this equation, we get $x^2 - x - 1 = 0$. By the quadratic formula,

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

If $x = \dfrac{1+\sqrt{5}}{2}$, then $x - \dfrac{1}{x} = 1$ by construction, and we have

$$\begin{aligned} f(1) &= \left(\frac{1+\sqrt{5}}{2}\right)^3 - \left(\frac{2}{1+\sqrt{5}}\right)^3 \\ &= \frac{1 + 3\sqrt{5} + 3\sqrt{5}^2 + \sqrt{5}^3}{8} - \frac{8}{1 + 3\sqrt{5} + 3\sqrt{5}^2 + \sqrt{5}^3} \\ &= \frac{16 + 8\sqrt{5}}{8} - \frac{8}{16 + 8\sqrt{5}} \\ &= 2 + \sqrt{5} - \frac{1}{2+\sqrt{5}} \\ &= 2 + \sqrt{5} - \frac{2-\sqrt{5}}{4-5} \\ &= 2 + \sqrt{5} + 2 - \sqrt{5} = 4 \end{aligned}$$

A similar calculation shows that if $x = \dfrac{1-\sqrt{5}}{2}$, then $x^3 - \dfrac{1}{x^3} = 4$ as well. Therefore $f(1) = 4$.

Answer: 4

20. We will begin by counting the total number of 1s that are printed when the integers from 1 through 99 are printed.
The integers less than 100 that have at least one 1 are

$$1, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 21, 31, 41, 51, 61, 71, 81, 91$$

and a total of 20 of the digits written above are 1.
When printing the integers from 100 through 199, there will be 100 1s coming from the hundreds digit of each of these integers, plus an additional 20 1s coming from the tens and units digits, as counted above.
When printing the integers from 200 to 299, the digit 1 will be printed 20 times.
Since the 100s digit is not 1 again from 300 to 999, the digit 1 will be printed exactly 20 times from 300 to 399, 400 to 499, and so on up to 900 to 999.
Therefore, the number of times the digit 1 is printed when the integers from 1 through 999 are printed is

$$9 \times 20 + 120 = 300$$

When printing the integers from 1000 to 1099, the digit 1 will be printed once as the thousands digit for each integer, and an additional 20 times for units and tens digits, for a total of 120 times. Adding to the previous total, in printing the first 1099 positive integers, the computer prints the digit 1 a total of $300 + 120 = 420$ times.
When the computer prints the integers from 1100 to 1199, there will be two 1s for each integer coming from the thousands and hundreds digits, plus an additional 20 1s, for a total of 220 1s. The computer will have printed $420 + 220 = 640$ 1s when it has printed the first 1199 positive integers.
The number of 1s from 1200 to 1299 is 120, and the number of 1s from 1300 through 1399 is also 120, so after the first 1399 positive integers have been printed, $640 + 120 + 120 = 880$ 1s have been printed.
There are exactly 120 1s from 1400 to 1499, for a total of $880 + 120 = 1000$ by the time 1499 is printed.
Since the integer 1499 has one of its digits equal to 1, the 1000^{th} 1 is printed when the integer 1499 is printed.

ANSWER: 1499

21. In this solution, logarithms are base 10 unless their base is explicitly given.
We can rewrite x as $x = 2024^{\log_{2024} x}$. Using this, exponent rules, and the change of base formula for logarithms, we get

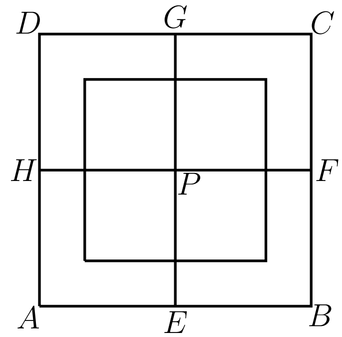
$$\begin{aligned}
4\sqrt{506} &= x^{\log 2024} + 2024^{\log x} \\
&= \left(2024^{\log_{2024} x}\right)^{\log 2024} + 2024^{\log x} \\
&= 2024^{\frac{\log x}{\log 2024} \cdot \log 2024} + 2024^{\log x} \\
&= 2024^{\log x} + 2024^{\log x}
\end{aligned}$$

Therefore, we get $4\sqrt{506} = 2 \cdot 2024^{\log x}$ or $2\sqrt{506} = 2024^{\log x}$.
Noting that $2\sqrt{506} = 2024^{\frac{1}{2}}$, we conclude that $\log x = \frac{1}{2}$. Therefore, $x = 10^{\frac{1}{2}}$ or $x = \sqrt{10}$.

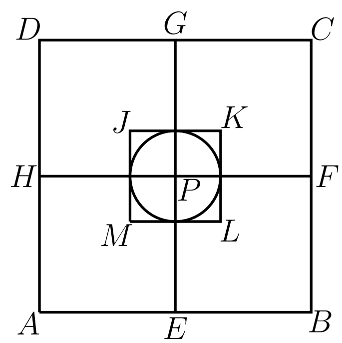
ANSWER: $\sqrt{10}$

22. The unit circle's position is completely determined by the location of its centre. Let R_1 be the region inside the square with the property that if the centre of the circle is inside R_1, then the

circle will be completely inside the square. Similarly, let R_2 be the region inside the square with the property that if the centre of the circle is inside R_2, then the circle will intersect exactly two of the line segments. The probability is the area of R_2 divided by the area of R_1.
In the diagram below, a square of side length 4 is centred inside $ABCD$ so that there is a "strip" of uniform width 1 between the squares.



Notice that every point inside the smaller square is at least 1 unit away from the boundary of $ABCD$, and every point inside the "strip" is less than 1 unit away from the boundary of $ABCD$. Therefore, a circle of radius 1 will be completely inside $ABCD$ exactly when its centre is inside the smaller square.
This means the smaller square is R_1, so the area of R_1 is $4 \times 4 = 16$.
In the diagram below, a circle of radius 1 and a square of side-length 2 are centred at P. The vertices of the square are labelled J, K, L, and M.



Suppose the centre of the circle is outside the square of side-length 2.
If the centre is somewhere above JK, then every point on EP, FP, and HP is more than one unit away from the centre of the unit circle, so it is impossible for the unit circle to intersect any of these line segments.
Therefore, all points above JK are outside of R_2.
Similarly, all points below LM, to the left of JM, and to the right of KL are outside of R_2.
These four regions combine to form the entire region outside $JKLM$, so we conclude that R_2 is contained in R_2.
If the centre of the unit circle is inside the circle centred at P, then P is inside the unit circle (since both circles have radius 1). In this situation, all four of EP, FP, GP, and HP intersect the unit circle.
Therefore, the region inside the circle centred at P is *not* in R_2.
Finally, we suppose the centre of the unit circle is in one of the four regions inside $JKLM$ but outside the circle centred at P.

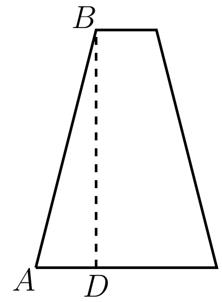
For example, suppose the centre of the unit circle is in the top-left of these four regions, with J as a vertex.
Then the centre of the circle is within 1 unit of HP and GP, so it must intersect both of them. However, every point on the interior FP and EP is more than 1 unit away from the centre of the unit circle (since all such points are at least as far away as P).
We have now shown that if the centre of the unit circle is in this small region, then the unit circle will intersect the interior of GP and HP, but not EP or FP.
By similar reasoning, the entire region inside $JKLM$ and outside the circle centred at P is inside R_2. We have also shown that these are the only possible points in R_2, so the area of R_2 is the difference between the area of $JKLM$ and a circle of radius 1, or $2 \times 2 - \pi = 4 - \pi$.
The probability is $\dfrac{4-\pi}{4^2} = \dfrac{4-\pi}{16}$.

ANSWER: $\frac{4-\pi}{16}$

23. The number of ways to place zero heads in a row is 1. This can only be done by placing six tails in the row.
The number of ways to place two heads in a row is $\dbinom{6}{2} = 15$.
The number of ways to place four heads in a row is $\dbinom{6}{4} = 15$.
The number of ways to place six heads in a row is 1.
Thus, the number of ways to place an even number of heads in a row is $1+15+15+1 = 32 = 2^5$.
Suppose the first five rows have been arranged to have an even number of heads in each row. This can be done in $(2^5)^5 = 2^{25}$ ways.
In order to have an even number of heads in a given column, there is only one possibility for the coin in the bottom row in that column: if the first five rows have an even number of heads in that column, then the sixth must be a tail, and if the first five rows have an odd number of heads in that column, then the sixth must be a head.
By the previous paragraph, there is exactly one way to assign coins to the bottom row so that the columns all have an even number of heads.
Once the coins in the bottom row are arranged, there are an even number of heads in each column, so the total number of heads in the grid must be even.
Since the number of heads in the first five rows is even and the total in all six rows is even, the number of heads in the last row must also be even.
In other words, if the first five rows are arranged to have an even number of heads in each row, then there is exactly one way to arrange the coins in the sixth row so that all conditions are satisfied. Therefore, the number of ways that the coins can be arranged is $\left(2^5\right)^5 = 2^{25}$.

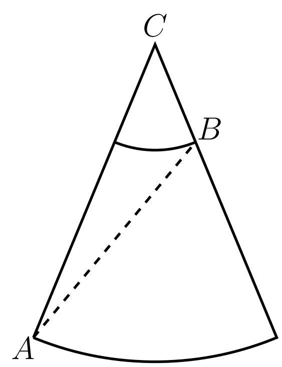
ANSWER: 2^{25}

24. The radius of a circle is equal to its circumference divided by 2π. Therefore, the radius of the base is $\dfrac{3\pi}{2\pi} = \dfrac{3}{2}$, and the radius of the horizontal cross-section through B is $\dfrac{\pi}{2\pi} = \dfrac{1}{2}$.
We will first determine the distance from point A to B directly along the surface. This is equal to the length along the slant height from the base to the cross-section.
We will draw a perpendicular line from B to the base of the cone, intersecting the base at D. The diagram below is a vertical cross -section of the bottom of the cone.



The length of BD is given to be $3\sqrt{7}$ and the length of AD is equal to the difference of the radii of the base and the cross-section, which is $\frac{3}{2} - \frac{1}{2} = 1$.

By the Pythagorean theorem, $AB = \sqrt{(3\sqrt{7})^2 + 1} = \sqrt{63+1} = 8$.
If we let E be the centre of the base of the cone, then $\triangle AEC$ is similar to $\triangle ADB$.
Since AE is a radius of the base, we have $AE = \frac{3}{2}$. We saw earlier that $AD = 1$, so $\frac{AE}{AD} = \frac{3}{2}$.
Since $\triangle AEC$ is similar to $\triangle ADB$, $\frac{AC}{AB} = \frac{AE}{AD} = \frac{3}{2}$, so $AC = \frac{3}{2}AB = \frac{3}{2}(8) = 12$.
We will now imagine cutting the top of the cone along line segment AC. This will give a sector of a circle with the circumference of the base forming the outer arc, and the circumference of the cross-section through B forming a smaller sector with the same angle and centre as the larger sector.



The radius of the larger sector is $AC = 12$, and the length of the arc at the bottom is equal to the circumference of the base of the cone, which is 3π.
The circumference of a circle with radius 12 is $2 \times 12 \times \pi = 24\pi$, and since $\frac{3\pi}{24\pi} = \frac{1}{8}$, the angle of the sector is $\frac{1}{8}$ of the angle in a full circle, or $\frac{1}{8} \times 360° = 45°$.
The length of BC is $12 - 8 = 4$, and the shortest path around the surface of the cone is the line segment connecting A to B in the sector.
In $\triangle ABC$ from the diagram above, we have $AC = 12$, $BC = 4$, and $\angle ACB = 45°$.
Using the Cosine law, we have

$$\begin{aligned}
p^2 &= AB^2 \\
&= AC^2 + BC^2 - 2(AC)(AB)\cos\angle ACB \\
&= 12^2 + 4^2 - 2(12)(4)\frac{1}{\sqrt{2}} \\
&= 160 - 48\sqrt{2}
\end{aligned}$$

Answer: $160 - 48\sqrt{2}$

25. The polynomial factors as

$$x^4 + 2x^3 + (3 - a^2)x^2 + (2 - 2a^2)x + (1 - a^2) = [x^2 + (1 - a)x + (1 - a)][x^2 + (1 + a)x + (1 + a)]$$

To find this factorization, you might first guess that the quartic factors as the product of two quadratics. The leading coefficient is 1 and the constant term factors as $1 - a^2 = (1 - a)(1 + a)$. Looking at other coefficients, the factorization can be discovered by trial and error.
The roots of $x^2 + (1 - a)x + (1 - a)$ are

$$\frac{a - 1 \pm \sqrt{(1 - a)^2 - 4(1 - a)}}{2} = \frac{a - 1 \pm \sqrt{a^2 + 2a - 3}}{2} = \frac{a - 1 \pm \sqrt{(a + 1)^2 - 4}}{2}$$

The roots of $x^2 + (1 + a)x + (1 + a)$ are

$$\frac{-(a + 1) \pm \sqrt{(a + 1)^2 - 4(a + 1)}}{2} = \frac{-(a + 1) \pm \sqrt{a^2 - 2a - 3}}{2} = \frac{-(a + 1) \pm \sqrt{(a - 1)^2 - 4}}{2}$$

A quadratic always has either 0, 1, or 2 real roots. Since the given quartic is the product of two quadratics, a root of the quartic must be a root of one or both of the quadratics.
For the quartic to have exactly 2 real roots, either one quadratic has 2 distinct real roots and the other has no real roots, or each quadratic has 1 real root, and those 2 roots are different.
Suppose both quadratics have exactly 1 real root. This implies both quadratics have their discriminant equal to 0.
Therefore, $(a + 1)^2 - 4 = 0$ and $(a - 1)^2 - 4 = 0$. The first equation is equivalent to $a + 1 = \pm 2$ and the second is equivalent to $a - 1 = \pm 2$.
Therefore, the first quadratic has exactly 1 real root when $a = -3$ or $a = 1$, and the second has exactly 1 real root when $a = -1$ or $a = 3$.
Therefore, there is no real number a for which both quadratics have exactly 1 real root.
We have now shown that the only way for the quartic to have exactly 2 real roots is for one of the quadratics to have 2 real roots and the other to have no real roots.
Examining discriminants again, we require either $(a + 1)^2 - 4 > 0$ and $(a - 1)^2 - 4 < 0$ or $(a + 1)^2 - 4 < 0$ and $(a - 1)^2 - 4 > 0$.
Suppose $(a + 1)^2 - 4 > 0$ and $(a - 1)^2 - 4 < 0$.
The second of these two inequalities is equivalent to $(a - 1)^2 < 4$, which is equivalent to $-2 < a - 1 < 2$ or $-1 < a < 3$.
The first is equivalent to $(a + 1)^2 > 4$, which is equivalent to $a + 1 < -2$ or $a + 1 > 2$. These inequalities can be rearranged to get $a < -3$ or $a > 1$.
We now conclude that the quartic will have exactly 2 real roots if $-1 < a < 3$ and either $a < -3$ or $a > 1$.
The inequality $a < -3$ is incompatible with the condition $-1 < a < 3$, so we must have $-1 < a < 3$ and $a > 1$.
Putting these together, we get that $1 < a < 3$.
Now suppose $(a + 1)^2 - 4 < 0$ and $(a - 1)^2 - 4 > 0$. By similar reasoning to the previous case, the first of these inequalities implies $-3 < a < 1$ and the second implies either $a > 3$ or $a < -1$.
This leads to $-3 < a < -1$, but we only want positive a values, so the answer is that the quartic has exactly 2 roots when $1 < a < 3$.

Answer: $1 < a < 3$

Relay Problems

(Note: Where possible, the solutions to parts (b) and (c) of each relay are written as if the value of t is not initially known, and then t is substituted at the end.)

0. (a) Evaluating, $\dfrac{2+5\times 5}{3} = \dfrac{2+25}{3} = \dfrac{27}{3} = 9$.

   (b) The area of a triangle with base $2t$ and height $2t-6$ is $\dfrac{1}{2}(2t)(2t-6)$ or $t(2t-6)$.
   The answer to (a) is 9, so $t = 9$ which means $t(2t-6) = 9(12) = 108$.

   (c) Since $\triangle ABC$ is isosceles with $AB = BC$, it is also true that $\angle BCA = \angle BAC$.
   The angles in a triangle add to $180°$, so

$$\begin{aligned}180° &= \angle ABC + \angle BAC + \angle BCA \\ &= \angle ABC + 2\angle BAC \\ &= t° + 2\angle BAC\end{aligned}$$

   The answer to (b) is 108, so $t = 108$. Therefore,

$$\angle BAC = \frac{1}{2}(180° - t°) = \frac{1}{2}(180° - 108°) = \frac{1}{2}(72°) = 36°.$$

Answer: $(9, 108, 36°)$

1. (a) The integers in the range that are divisible by 2 are $2, 4, 6, 8, 10, 12, 14, 16, 18, 20$.
   The integers in the range that are divisible by 3 are $3, 6, 9, 12, 15, 18$.
   The integers that are in both lists are 6, 12, and 18.
   Therefore, the integers that are in exactly one of the two lists are

$$2, 3, 4, 8, 9, 10, 14, 15, 16, 20$$

   There are 10 integers in the list, so the answer is 10.

   (b) Subtracting the second equation from the first, we get

$$\begin{aligned}(4x+3y) - (-4x+3y) &= 60 - (t+2) \\ 8x &= 58 - t\end{aligned}$$

   or $x = \dfrac{58-t}{8}$. Substituting $t = 10$ gives $x = \dfrac{58-10}{8} = \dfrac{48}{8} = 6$.

   (c) The surface area of a rectangular prism with dimensions a by b by c is $2(ab+ac+bc)$.
   The surface area of the rectangular prism in the problem is

$$2(6r\cdot r + r\cdot t + t\cdot 6r) = 2(6r^2 + 7rt) = 12r^2 + 14rt$$

   We are given that the surface area is $18r^2$, and so $18r^2 = 12r^2 + 14rt$ or $6r^2 = 14rt$.
   Since $r > 0$, we can divide by r to get $6r = 14t$ so $r = \dfrac{14t}{6} = \dfrac{7t}{3}$.
   Substituting $t = 6$ gives $r = \dfrac{7(6)}{3} = 14$.

Answer: $(10, 6, 14)$

2. (a) Rearranging the equation, we get $\frac{13}{2}y = -13x + 9$.
Multiplying through by $\frac{2}{13}$ gives $y = -2x + \frac{18}{13}$.
The slope of this line is -2, so a perpendicular line must have a slope equal to the negative reciprocal of -2, which is $\frac{-1}{-2} = \frac{1}{2}$.

(b) It is given that $AE = 4$ and $\frac{EB}{AE} = 2$, so $EB = 2AE = 2 \times 4 = 8$.
Therefore, $AB = AE + EB = 4 + 8 = 12$.
Since EF is parallel to BC, $\angle AFE = \angle ACB$ and $\angle AEF = \angle ABC$.
Therefore, $\triangle AEF$ and $\triangle ABC$ are similar by angle-angle similarity.
Since these triangles are similar, we get $\frac{AF}{AC} = \frac{AE}{AB} = \frac{4}{12} = \frac{1}{3}$.
Rearranging $\frac{AF}{AC} = \frac{1}{3}$ gives $3AF = AC$.
We also have that $AF = AC - FC = AC - 10$, so we can substitute to get $3(AC - 10) = AC$, which can be rearranged to get $2AC = 30$ or $AC = 15$.
Using that $\triangle AEF$ is similar to $\triangle ABC$, we have $\frac{BC}{EF} = \frac{AB}{AE} = \frac{12}{4} = 3$.
Rearranging $\frac{BC}{EF} = 3$ gives $BC = 3EF = 3(4t) = 12t$.
We have $AB = 12$, $AC = 15$, and $BC = 12t$, so the perimeter of $\triangle ABC$ is equal to $12 + 15 + 12t = 27 + 12t$.
Substituting $t = \frac{1}{2}$ gives $27 + 12 \times \frac{1}{2} = 27 + 6 = 33$.

(c) The parabola passes through the points with coordinates $(4, 5)$ and $(0, 5)$.
Since these points have the same y-coordinate, the axis of symmetry of the parabola must be at the average of the x-coordinates, or at $x = \frac{4+0}{2} = 2$.
The minimum value of the function occurs on the axis of symmetry, so the parabola achieves its minimum at $(2, -3)$.
This all implies that the parabola has an equation of the form $y = a(x-2)^2 - 3$ for some real number a.
Using that the parabola passes through $(4, 5)$, we get $5 = a(4-2)^2 - 3$ or $5 = 4a - 3$.
Solving this equation for a gives $a = 2$, so the parabola has equation $y = 2(x-2)^2 - 3$.
The parabola also passes through $\left(\frac{2t}{11}, h\right)$, which simplifies to $(6, h)$ when $t = 33$ is substituted.
Therefore, $h = 2(6-2)^2 - 3 = 2(4)^2 - 3 = 29$.

Answer: $(\frac{1}{2}, 33, 29)$

3. (a) The sum of the other 9 integers is $83 - 11 = 72$. Since there are 9 other integers, their average is $\frac{72}{9} = 8$.

(b) Since the y-intercept is -2, the line passes through the point $(x, y) = (0, -2)$.
Substituting $(x, y) = (0, -2)$ and $(x, y) = (16, 2)$ into the equation $ax + by = t$ gives

$$\begin{aligned} -2b &= t \\ 16a + 2b &= t \end{aligned}$$

The first equation is equivalent to $b = -\dfrac{t}{2}$, and if we add the equations we get $16a = 2t$ or $a = \dfrac{t}{8}$.

Therefore, $a - b = \dfrac{t}{8} - \left(-\dfrac{t}{2}\right) = \dfrac{5t}{8}$.

Substituting $t = 8$ gives $a - b = 5$.

(c) A square with side length $(t + 2)$ has area $(t + 2)^2$.
The rectangle has area $(4t + 8)(t^3 + 2t^2)$, which can be factored as $4(t + 2)t^2(t + 2)$.
The number of squares needed is the area of the rectangle divided by the area of the squares, or

$$\frac{4(t + 2)t^2(t + 2)}{(t + 2)^2} = 4t^2$$

Since $t = 5$, the number of squares needed is $4(5)^2 = 100$.

Answer: $(8, 5, 100)$