The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2024 Canadian Senior Mathematics Contest

Wednesday, November 13, 2024
(in North America and South America)

Thursday, November 14, 2024
(outside of North America and South America)

Solutions

Part A

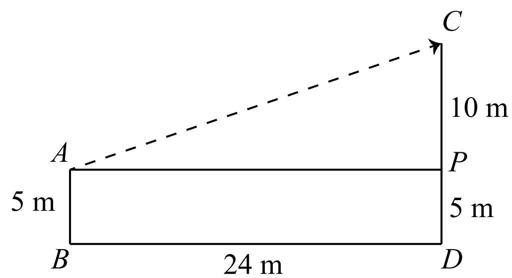
1. *Solution 1*
   Evaluating, $\sqrt{10^2 + 2 \cdot 10 \cdot 11 + 11^2} = \sqrt{100 + 220 + 121} = \sqrt{441} = 21$.

   Solution 2
   Since $x^2 + 2xy + y^2 = (x + y)^2$, then

$$\sqrt{10^2 + 2 \cdot 10 \cdot 11 + 11^2} = \sqrt{(10 + 11)^2} = \sqrt{21^2} = 21$$

ANSWER: 21

2. Label the shorter tree as AB and the taller tree as CD, as shown.
   Draw a horizontal line from A to P on CD.



   Since AB and PD are vertical and AP and BD are horizontal, $ABDP$ is a rectangle which means that $PD = 5$ m and $AP = 24$ m.
   Since $CD = 15$ m and $PD = 5$ m, then $CP = CD - PD = 10$ m.
   By the Pythagorean Theorem,

$$AC = \sqrt{AP^2 + CP^2} = \sqrt{(24 \text{ m})^2 + (10 \text{ m})^2} = \sqrt{676 \text{ m}^2} = 26 \text{ m}$$

   since $AC > 0$.
   Since the bird flies 26 m at 4 m/s, its time to complete the flight is $\dfrac{26 \text{ m}}{4 \text{ m/s}} = 6.5$ s.

ANSWER: 6.5 s

3. Since Team Why had scored 3 goals at the end of the game, then $y \leq 3$.
   Since Team Zed had scored 2 goals at the end of the game, then $z \leq 2$.
   Therefore, $0 \leq y \leq 3$ and $0 \leq z \leq 2$.
   The only additional restriction is that y and z are both integers.
   Thus, there are 4 possible values for y and 3 possible values for z, which means that there are $4 \cdot 3 = 12$ possible pairs (y, z).
   These pairs are

$$(y, z) = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2)$$

   These pairs correspond to the possible scores at half-time.

ANSWER: 12

4. We are told that a, b, c, d are positive integers with $abc = d$ and $d \leq 8$.
We determine the number of possible quadruples (a, b, c, d) by looking at each possible value of d.

If $d = 1$, then $abc = 1$. Since a, b, c are positive integers, then $a = b = c = 1$. This means that there is 1 quadruple in this case, namely $(1, 1, 1, 1)$.

If $d = 2$, then $abc = 2$. Since 2 is prime, then one of a, b and c equals 2 and the other two variables equal 1.
This gives 3 quadruples: $(a, b, c, d) = (2, 1, 1, 2), (1, 2, 1, 2), (1, 1, 2, 2)$.
Similarly, when $d = 3$, $d = 5$ and $d = 7$ (the other possible prime values of d), there are 3 quadruples.

If $d = 4$, then $abc = 4$. We need to determine the ways in which 4 can be factored as the product of three positive integers.
These ways are $1 \cdot 1 \cdot 4 = 4$ and $1 \cdot 2 \cdot 2 = 4$.
In each case, there are 3 ways to arrange the factors on the left side, which are the possible values of a, b and c:

$$1 \cdot 1 \cdot 4 = 1 \cdot 4 \cdot 1 = 4 \cdot 1 \cdot 1 = 4$$

$$1 \cdot 2 \cdot 2 = 2 \cdot 1 \cdot 2 = 2 \cdot 2 \cdot 1 = 4$$

Thus, there are 6 quadruples in this case.

If $d = 6$, then $abc = 6$. The possible factorizations of 6 into three positive factors are $1 \cdot 1 \cdot 6 = 6$ and $1 \cdot 2 \cdot 3 = 6$.
In the first case, there are again 3 quadruples.
In the second case, there are 6 ways of arranging the factors:

$$1 \cdot 2 \cdot 3 = 1 \cdot 3 \cdot 2 = 2 \cdot 1 \cdot 3 = 2 \cdot 3 \cdot 1 = 3 \cdot 1 \cdot 2 = 3 \cdot 2 \cdot 1 = 6$$

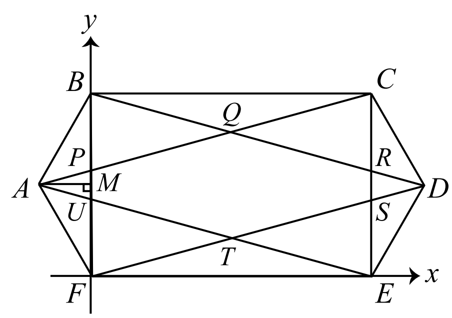
Thus, if $d = 6$, there are 9 quadruples.

If $d = 8$, then $abc = 8$. The possible factorizations of 8 into three positive factors are $1 \cdot 1 \cdot 8 = 8$ and $1 \cdot 2 \cdot 4 = 8$ and $2 \cdot 2 \cdot 2 = 8$.
These cases give 3, 6 and 1 quadruples, for a total of 10.

Combining all of the cases, there are $1 + 3 + 3 + 3 + 3 + 6 + 9 + 10 = 38$ quadruples.

ANSWER: 38

5. We use coordinate geometry.
Suppose that F is the origin and E lies on the positive x-axis.
The coordinates of F are $(0, 0)$. Since $FE = 6$, the coordinates of E are $(6, 0)$.
Next, we find the coordinates of B.
We note that $FA = AB = 2$.
Since the six interior angles of hexagon $ABCDEF$ are equal, then the measure of each is equal to $\frac{1}{6} \cdot 720° = 120°$. (The sum of the measures of the interior angles of any hexagon is $720°$.)
This means that $\triangle FAB$ is isosceles with $FA = AB = 2$ and $\angle FAB = 120°$.
Since $\angle FAB = 120°$, then $\angle AFB = \angle ABF = \frac{1}{2}(180° - 120°) = 30°$.
Since $\angle AFE = 120°$ and $\angle AFB = 30°$, then $\angle BFE = \angle AFE - \angle AFB = 90°$, which means that BF is vertical and so B lies on the positive y-axis.
If we draw a perpendicular from A to a point M on BF, then, because $\triangle FAB$ is isosceles, M is the midpoint of BF and AM bisects $\angle FAB$.



This means that $\triangle BAM$ is a 30°-60°-90° triangle.
Thus, $BM = \frac{\sqrt{3}}{2}AB = \sqrt{3}$ and so $BF = 2\sqrt{3}$ which means that the coordinates of B are $(0, 2\sqrt{3})$.
(We could have determined the length of BF using the cosine law in $\triangle BAF$.)
Also, $AM = \frac{1}{2}AB = 1$. This means that the coordinates of A are $(-1, \sqrt{3})$.
Using similar arguments, the coordinates of C are $(6, 2\sqrt{3})$ and the coordinates of D are $(7, \sqrt{3})$.
We can determine the area of hexagon $PQRSTU$ by starting with the area of rectangle $BCEF$ (which is $6 \cdot 2\sqrt{3} = 12\sqrt{3}$) and subtracting the areas of $\triangle UFT$, $\triangle BPQ$, $\triangle CRQ$, $\triangle SET$, $\triangle BCQ$, and $\triangle FET$.
By symmetry, the areas of $\triangle UFT$, $\triangle BPQ$, $\triangle CRQ$, $\triangle SET$ are all equal.
Also, by symmetry, the areas of $\triangle BCQ$ and $\triangle FET$ are equal.
We determine the area of $\triangle UFT$ and the area of $\triangle FET$.
To do this, we determine the coordinates of U and of T.
The line segment FD passes through $F(0, 0)$ and $D(7, \sqrt{3})$.
Thus, it has slope $\frac{\sqrt{3}}{7}$ and equation $y = \frac{\sqrt{3}}{7}x$.
The line segment AE passes through $A(-1, \sqrt{3})$ and $E(6, 0)$.
Thus, it has slope $\frac{\sqrt{3}}{-7}$ and equation $y = -\frac{\sqrt{3}}{7}(x - 6)$ or $y = -\frac{\sqrt{3}}{7}x + \frac{6\sqrt{3}}{7}$.
This means that the coordinates of U (which is the y-intercept of this line) are $(0, \frac{6\sqrt{3}}{7})$.
To find the coordinates of T, we find the point of intersection of AE and FD.
Equating expressions for y, we obtain $\frac{\sqrt{3}}{7}x = -\frac{\sqrt{3}}{7}x + \frac{6\sqrt{3}}{7}$ which gives $\frac{2\sqrt{3}}{7}x = \frac{6\sqrt{3}}{7}$ or $x = 3$. (It should not be surprising that the x-coordinate of T is $x = 3$.)
When $x = 3$, we obtain $y = \frac{3\sqrt{3}}{7}$, and so the coordinates of T are $(3, \frac{3\sqrt{3}}{7})$.
So $\triangle FET$ has base $FE = 6$ and height equal to the distance from T to FE (which is $\frac{3\sqrt{3}}{7}$).
Thus, the area of $\triangle FET$ is $\frac{1}{2} \cdot 6 \cdot \frac{3\sqrt{3}}{7} = \frac{9\sqrt{3}}{7}$.

Also, $\triangle UFT$ has vertical base UF (which has length $\frac{6\sqrt{3}}{7}$) and horizontal height equal to the distance from T to UF (which is 3). Thus, the area of $\triangle UFT$ is $\frac{1}{2} \cdot \frac{6\sqrt{3}}{7} \cdot 3 = \frac{9\sqrt{3}}{7}$. (Can you see a reason why this should be equal to the area of $\triangle FET$?)
Finally, this means that the area of $PQRSTU$ is equal to $12\sqrt{3} - 6 \cdot \frac{9\sqrt{3}}{7}$ or $\frac{84\sqrt{3}}{7} - \frac{54\sqrt{3}}{7}$ which equals $\frac{30\sqrt{3}}{7} = \frac{\sqrt{2700}}{7}$.
Therefore, $(n, t) = (2700, 7)$.

ANSWER: $(2700, 7)$

6. A list of m distinct positive integers has a sum that is at least $1 + 2 + \cdots + (m-1) + m$ which is equal to $\frac{1}{2}m(m+1)$.
We note that $\frac{1}{2} \cdot 63 \cdot 64 = 2016$ and $\frac{1}{2} \cdot 64 \cdot 65 = 2080$.
In other words, the sum $1 + 2 + \cdots + 63 + 64$ is greater than 2024 and any sum of more than 64 distinct positive integers is greater still.
This means that a Gleeson list consists of at most 63 integers.
Note that $1 + 2 + \cdots + 62 + 63 = \frac{1}{2} \cdot 63 \cdot 64 = 2016$ and so $1 + 2 + \cdots + 62 + 63 + 8 = 2024$ which means that $1 + 2 + \cdots + 61 + 62 + 71 = 2016$.
This means that there is at least one Gleeson list of length 63 and there cannot be a Gleeson list of length greater than 63, so M, the maximum length of a Gleeson list, is 63.
We need to determine the number of Gleeson lists of length $M = 63$.
Suppose that $a_1, a_2, a_3, \ldots, a_{62}, a_{63}$ is a Gleeson list.
That is, $a_1, a_2, a_3, \ldots, a_{62}, a_{63}$ are positive integers with $a_1 < a_2 < \cdots < a_{62} < a_{63}$ and

$$a_1 + a_2 + \cdots + a_{62} + a_{63} = 2024$$

For each j with $1 \leq j \leq 63$, let $b_j = a_j - j$, which means that $a_j = j + b_j$.
That is, we write a generic Gleeson list as

$$1 + b_1, 2 + b_2, \ldots, 62 + b_{62}, 63 + b_{63}$$

where the b_j's are the "extra" in each term.
For each j with $1 \leq j \leq 62$, since a_j and a_{j+1} are integers with $a_{j+1} > a_j$, then $a_{j+1} \geq a_j + 1$.
This means that $j + 1 + b_{j+1} \geq j + b_j + 1$ or $b_{j+1} \geq b_j$.
This means that $b_1, b_2, \ldots, b_{62}, b_{63}$ is a list of integers with $b_1 \leq b_2 \leq \cdots \leq b_{62} \leq b_{63}$.
Since $a_1 = 1 + b_1 \geq 1$, then $b_1 \geq 0$ and so $0 \leq b_1 \leq b_2 \leq \cdots \leq b_{62} \leq b_{63}$.
Also, since $a_1 + a_2 + \cdots + a_{62} + a_{63} = 2024$ then

$$(1 + b_1) + (2 + b_2) + \cdots + (62 + b_{62}) + (63 + b_{63}) = 2024$$

Since $1 + 2 + \cdots + 62 + 63 = 2016$, then

$$b_1 + b_2 + \cdots + b_{62} + b_{63} = 8$$

In other words, to count the number of Gleeson lists of length 63, we need to count the number of non-decreasing lists of non-negative integers $b_1, b_2, \ldots, b_{62}, b_{63}$ with a sum of 8.
We determine these lists by categorizing them using the number of non-zero terms, in each case with the understanding that the terms before these are all 0.

- 1 non-zero term: 8
- 2 non-zero terms: 1, 7; 2, 6; 3, 5; 4, 4

- 3 non-zero terms: 1, 1, 6; 1, 2, 5; 1, 3, 4; 2, 2, 4; 2, 3, 3
- 4 non-zero terms: 1, 1, 1, 5; 1, 1, 2, 4; 1, 1, 3, 3; 1, 2, 2, 3; 2, 2, 2, 2
- 5 non-zero terms: 1, 1, 1, 1, 4; 1, 1, 1, 2, 3; 1, 1, 2, 2, 2
- 6 non-zero terms: 1, 1, 1, 1, 1, 3; 1, 1, 1, 1, 2, 2
- 7 non-zero terms: 1, 1, 1, 1, 1, 1, 2
- 8 non-zero terms: 1, 1, 1, 1, 1, 1, 1, 1

In each case, we determine the lists by starting with the largest possible entries at the right, and adjusting to the left in a systematic way.
There are 22 such lists, and so 22 Gleeson lists of length 63.

ANSWER: 22

Part B

1. (a) Since $64 = 2^6$, then $2^{x+1} = 2^6$ which gives $x + 1 = 6$ and so $x = 5$.

   (b) Comparing exponents in the two equations, we obtain $t + u = 8$ and $u - 3 = 2$.
   From the second equation, $u = 5$.
   Substituting $u = 5$ into $t + u = 8$ gives $t = 3$.
   Therefore, $t = 3$ and $u = 5$.

   (c) Since m and r are integers, we can compare exponents to obtain $2m+r = 7$ and $3m-r = 3$. (We can do this because every positive integer greater than 1 can be uniquely factored as a product of prime numbers.)
   Adding these two equations, we obtain $(2m + r) + (3m - r) = 7 + 3$ or $5m = 10$, which gives $m = 2$.
   Substituting into the first equation gives $2 \cdot 2 + r = 7$ and so $r = 3$.
   Therefore, $m = 2$ and $r = 3$.

   (d) *Solution 1*
   First, we note that $80 = 16 \cdot 5 = 2^4 5^1$.
   Since $2^{p+q}5^{p-q} = 2^4 5^1$ and p and q are integers, then comparing exponents, we obtain $p + q = 4$ and $p - q = 1$.
   Adding these equations, we obtain $2p = 5$ which gives $p = 2.5$.
   This means that there are no integers p and q that satisfy this equation.

   Solution 2
   First, we note that $80 = 16 \cdot 5 = 2^4 5^1$.
   This means that 80 includes exactly one factor of 5.
   Since $2^{p+q}5^{p-q} = 80$, then the left side also contains only one factor of 5.
   This means that $p - q = 1$, which means that p and q are either odd and even, or even and odd.
   This in turn means that $p + q$ is the sum of an even integer and an odd integer, so is odd.
   But 80 includes an even number of factors of 2 and we know that the left side includes an odd number of factors of 2.
   This is a contradiction, and so there are no integers p and q that satisfy this equation.

2. (a) *Solution 1*
   Using the quadratic formula, the solutions of the quadratic equation $x^2 - 2x - 1 = 0$ are

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-1)}}{2} = \frac{2 \pm \sqrt{8}}{2} = \frac{2 \pm 2\sqrt{2}}{2} = 1 \pm \sqrt{2}$$

   We set $r = 1 + \sqrt{2}$ and $s = 1 - \sqrt{2}$.
   Then $2r + s = 2(1+\sqrt{2}) + (1-\sqrt{2}) = 3+\sqrt{2}$ and $r + 2s = (1+\sqrt{2}) + 2(1-\sqrt{2}) = 3-\sqrt{2}$.
   Therefore, a quadratic equation that has $x = 3 + \sqrt{2}$ and $x = 3 - \sqrt{2}$ as solutions is

$$(x - (3 + \sqrt{2}))(x - (3 - \sqrt{2})) = 0$$

   or

$$x^2 - ((3 + \sqrt{2}) + (3 - \sqrt{2}))x + (3 + \sqrt{2})(3 - \sqrt{2}) = 0$$

   Since $(3 + \sqrt{2})(3 - \sqrt{2}) = 3^2 - (\sqrt{2})^2 = 9 - 2 = 7$, this simplifies to $x^2 - 6x + 7 = 0$.
   Thus, $b = -6$ and $c = 7$.

Solution 2

If the quadratic equation $x^2 + Bx + C = 0$ has solutions $x = r$ and $x = s$, then the quadratic expressions x^2+Bx+C and $(x-r)(x-s)$ must be the same because both have a leading coefficient of 1, which means that

$$x^2 + Bx + C = x^2 - rx - sx + rs = x^2 - (r+s)x + rs$$

and so $r + s = -B$ and $rs = C$.

Here, the quadratic equation $x^2 - 2x - 1 = 0$ has roots $x = r$ and $x = s$, which means that $r + s = 2$ and $rs = -1$.

In this case,

$$\begin{aligned}(2r+s)+(r+2s) &= 3r+3s = 3(r+s) = 3\cdot 2 = 6\\ (2r+s)(r+2s) &= 2r^2+5rs+2s^2\\ &= 2r^2+4rs+2s^2+rs\\ &= 2(r^2+2rs+s^2)+rs\\ &= 2(r+s)^2+rs\\ &= 2\cdot 2^2+(-1) = 7\end{aligned}$$

Therefore, a quadratic equation with roots $x = 2r + s$ and $x = r + 2s$ is $x^2 - 6x + 7 = 0$. Thus, $b = -6$ and $c = 7$.

(b) *Solution 1*

Suppose that the polynomial $f(x) = x^2+mx+p$ has two distinct positive real roots $x = r$ and $x = s$.

This means that $r + s = -m$ and $rs = p$, as shown in part (a) Solution 2).

In this case, $m^2 - 2p = (-m)^2 - 2p = (r+s)^2 - 2rs = r^2 + 2rs + s^2 - 2rs = r^2 + s^2$ and $p^2 = (rs)^2 = r^2s^2$.

This means that we can rewrite $g(x)$ as $g(x) = x^2 - (r^2 + s^2)x + r^2s^2$ which factors as $g(x) = (x-r^2)(x-s^2)$. (Can you see why this is true by looking at the sum and product of roots?)

Since $g(x) = (x-r^2)(x-s^2)$, then $g(x)$ has roots r^2 and s^2.

We recall that r and s are positive and distinct.

Since neither equals 0, then r^2 and s^2 are positive.

Since $r \neq \pm s$, then r^2 and s^2 are distinct.

This means that $g(x)$ has two distinct positive real roots.

Solution 2

We proceed by using the sum and product of roots as shown in part (a) Solution 2.

Suppose that the roots of $f(x) = x^2 + mx + p$ are $x = r$ and $x = s$.

Since r and s are positive, then $r + s > 0$ and $rs > 0$.

Since $r + s = -m$, then $m < 0$. Since $rs = p$, then $p > 0$.

Since $f(x)$ has two distinct real roots, then its discriminant is positive.

In particular, $m^2 - 4p > 0$.

Now, the discriminant of $g(x) = x^2 - (m^2 - 2p)x + p^2$ is

$$\begin{aligned}\Delta &= (m^2 - 2p)^2 - 4(1)(p^2) \\ &= m^4 - 4m^2p + 4p^2 - 4p^2 \\ &= m^4 - 4m^2p \\ &= m^2(m^2 - 4p)\end{aligned}$$

Since $m^2 > 0$ (because $m < 0$) and $m^2 - 4p > 0$, then $\Delta = m^2(m^2 - 4p) > 0$.
This means that $g(x)$ has two distinct real roots.
Next, the sum of the (real) roots of $g(x)$ is $m^2 - 2p$ and their product is p^2.
We note that $m^2 - 2p = (m^2 - 4p) + 2p$ which is the sum of two positive real numbers and so is positive.
Also, p^2 is positive since $p > 0$.
Since the product of the roots of $g(x)$ is positive, the roots are either both positive or both negative.
Since the sum of the roots of $g(x)$ is positive, the roots cannot both be negative, and so both are positive.
In summary, $g(x)$ has two distinct positive real roots, as required.

(c) *Solution 1*
Suppose that a cubic equation $x^3 + Ax^2 + Bx + C = 0$ has roots $x = r$, $x = s$, and $x = t$.
As we saw in (a), this means that

$$\begin{aligned}x^3 + Ax^2 + Bx + C &= (x - r)(x - s)(x - t) \\ &= (x^2 - (r + s)x + rs)(x - t) \\ &= x^3 - (r + s)x^2 + rsx - tx^2 + (rt + st)x - rst \\ &= x^3 - (r + s + t)x^2 + (rs + rt + st)x - rst\end{aligned}$$

and so $A = -(r + s + t)$ and $B = rs + rt + st$ and $C = -rst$.
Also, if $A = -(r + s + t)$ and $B = rs + rt + st$ and $C = -rst$, then a cubic equation with roots r, s and t is $x^3 + Ax + Bx + C = 0$.

Consider the polynomial $f_1(x) = x^3 + A_1x^2 + B_1x + C_1 = x^3 - 6x^2 + 10x - 5$.
Since $f(1) = 1 - 6 + 10 - 5 = 0$, then $x = 1$ is a root.
Factoring, we obtain

$$f_1(x) = (x - 1)(x^2 - 5x + 5)$$

By the quadratic formula, the roots of $x^2 - 5x + 5 = 0$ are

$$x = \frac{5 \pm \sqrt{(-5)^2 - 4(1)(5)}}{2} = \frac{5 \pm \sqrt{5}}{2}$$

We note that $\dfrac{5 + \sqrt{5}}{2} \approx 3.62$ and $\dfrac{5 - \sqrt{5}}{2} \approx 1.38$.
This means that $f_1(x)$ has three distinct positive real roots.

Suppose that, for some positive integer $n \geq 2$, we have

$$\begin{aligned}A_{n-1} &= -(r + s + t) \\ B_{n-1} &= rs + rt + st \\ C_{n-1} &= -rst\end{aligned}$$

for some distinct, positive real numbers r, s and t.
Then

$$\begin{aligned}
A_n &= 2B_{n-1} - (A_{n-1})^2 \\
&= 2(rs + rt + st) - (-(r + s + t))^2 \\
&= 2(rs + rt + st) - (r + s + t)^2 \\
&= 2rs + 2rt + 2st - (r^2 + s^2 + t^2 + 2rs + 2rt + 2st) \\
&= -(r^2 + s^2 + t^2)
\end{aligned}$$

$$\begin{aligned}
B_n &= (B_{n-1})^2 - 2A_{n-1}C_{n-1} \\
&= (rs + rt + st)^2 - 2(-(r + s + t))(-rst) \\
&= r^2s^2 + r^2t^2 + s^2t^2 + 2r^2st + 2rs^2t + 2rst^2 - (2r^2st + 2rs^2t + 2rst^2) \\
&= r^2s^2 + r^2t^2 + s^2t^2
\end{aligned}$$

$$\begin{aligned}
C_n &= -(C_{n-1})^2 \\
&= -(-rst)^2 \\
&= -r^2s^2t^2
\end{aligned}$$

Consider the polynomial

$$f_n(x) = x^3 + A_nx^2 + B_nx + C_n = x^3 - (r^2 + s^2 + t^2)x^2 + (r^2s^2 + r^2t^2 + s^2t^2)x - r^2s^2t^2$$

This polynomial has roots $x = r^2$, $x = s^2$ and $x = t^2$, since the coefficients of the polynomial are the correct combinations of these roots.
This means that the roots of $f_n(x)$ are the squares of the roots of $f_{n-1}(x)$.
Here, $f_1(x)$ has three distinct positive real roots.
Since the roots of $f_2(x)$ are the squares of the roots of $f_1(x)$, then they are real, positive and distinct.
Continuing in this way, the roots of $f_3(x)$, $f_4(x)$, $\ldots$, $f_{100}(x)$ will each be the squares of the roots of the previous polynomial in the list, and so will be real, positive and distinct.
Therefore, the polynomial $f_{100}(x)$ has three distinct positive real roots.

Solution 2
From Solution 1, we know that the polynomial $f_1(x) = x^3 - 6x^2 + 10x - 5$ has three distinct positive real roots, one of which is equal to 1.
Suppose that A, B and C are integers and also that the polynomial $f(x) = x^3 + Ax^2 + Bx + C$ has three distinct positive real roots, one of which is 1.
Consider the polynomial $h(x) = x^3 + (2B - A^2)x^2 + (B^2 - 2AC)x - C^2$.
Note that the coefficients of $h(x)$ are integers because A, B and C are integers.
We show that $h(x)$ has three distinct positive real roots, one of which is equal to 1.

Since $f(x)$ has $x = 1$ as a root, then $f(1) = 1 + A + B + C = 0$ which gives $B = -1 - A - C$.
Factoring out $x - 1$ from $f(x)$, we obtain

$$f(x) = x^3 + Ax^2 + Bx + C = (x - 1)(x^2 + (A + 1)x - C)$$

To find the quadratic factor here, we write $f(x) = (x - 1)(x^2 + px + q)$, expand to obtain $x^3 + Ax^2 + Bx + C = x^3 + (p - 1)x^2 + (q - p)x - q$, and compare coefficients to get $C = -q$ (giving $q = -C$) and $p - 1 = A$ (giving $p = A + 1$).
We note that comparing coefficients of x gives $q - p = B$ or $-C - A - 1 = B$, which is

consistent with the relationship above.
Since $f(x)$ has three distinct positive real roots, one of which is equal to 1, then the quadratic $x^2 + (A+1)x - C$ has two distinct positive real roots neither of which equals 1. This means that $1 + (A+1) - C \neq 0$ and so $C \neq A+2$, a fact that we will use later.
Also, we can deduce the following:

- The product of its roots is positive; that is, $-C > 0$ or $C < 0$.
- The sum of its roots is positive; that is, $-A - 1 > 0$ or $A < -1$.
- The discriminant is positive; that is $(A+1)^2 + 4C > 0$.

To summarize, so far, we know that $B = -1 - A - C$ and $C < 0$ and $A < -1$ and $(A+1)^2 + 4C > 0$.

We now examine $h(x) = x^3 + (2B - A^2)x^2 + (B^2 - 2AC)x - C^2$.
We start by showing that $x = 1$ is a root of $h(x)$ by showing that $h(1) = 0$.
Now

$$\begin{aligned} h(1) &= 1 + (2B - A^2) + (B^2 - 2AC) - C^2 \\ &= B^2 + 2B + 1 - (A^2 + 2AC + C^2) \\ &= (B+1)^2 - (A+C)^2 \\ &= (B + 1 + A + C)(B + 1 - A - C) \end{aligned}$$

Recall that $B = -1 - A - C$ or $1 + A + B + C = 0$ and so $h(1) = 0$ and so $x = 1$ is a root of $h(x)$.
Factoring $h(x)$ in a similar manner to how we factored $f(x)$, we obtain

$$\begin{aligned} h(x) &= x^3 + (2B - A^2)x^2 + (B^2 - 2AC)x - C^2 \\ &= (x-1)(x^2 + (2B - A^2 + 1)x + C^2) \\ &= (x-1)(x^2 + (-2 - 2A - 2C - A^2 + 1)x + C^2) \\ &= (x-1)(x^2 - (A^2 + 2A + 2C + 1)x + C^2) \end{aligned}$$

We now need to show that the quadratic polynomial $q(x) = x^2 - (A^2 + 2A + 2C + 1)x + C^2$ has two distinct real positive roots neither of which equals 1.
First, we show that the roots of $q(x)$ are real and distinct by examining the discriminant:

$$\begin{aligned} \Delta &= (A^2 + 2A + 2C + 1)^2 - 4C^2 \\ &= (A^2 + 2A + 2C + 1 + 2C)(A^2 + 2A + 2C + 1 - 2C) \\ &= ((A+1)^2 + 4C)(A+1)^2 \end{aligned}$$

The first factor is positive since $(A+1)^2 + 4C$ is positive; the second factor is positive since $(A+1)^2$ is positive because $A \neq -1$.
Therefore, $\Delta > 0$ which means that $q(x)$ has two distinct real roots.
To show that $q(x)$ does not have 1 as a root, we suppose that $q(1) = 0$ and obtain a contradiction:

$$\begin{aligned} q(1) &= 0 \\ 1 - A^2 - 2A - 2C - 1 + C^2 &= 0 \\ C^2 - 2C + 1 - A^2 - 2A - 1+ &= 0 \\ (C-1)^2 - (A+1)^2 &= 0 \\ (C - 1 + A + 1)(C - 1 - A - 1) &= 0 \\ (C + A)(C - A - 2) &= 0 \end{aligned}$$

Since $C < 0$ and $A < 0$, then $C + A \neq 0$. Further, we saw above that $C \neq A + 2$. Together, this means that $q(1) \neq 0$.
This in turn means that $h(x)$ has three distinct real roots.
Finally, we need to show that the roots of $q(x)$ are positive.
Since $q(x) = x^2 - (A^2 + 2A + 2C + 1)x + C^2$ then the product of the two real roots of $q(x)$ is C^2 and the sum of the roots is $A^2 + 2A + 2C + 1$.
Since the product is C^2 and $C < 0$, then the product is positive, which means that both roots are positive or both are negative.
Since the sum of the roots is

$$A^2 + 2A + 2C + 1 = ((A + 1)^2 + 4C) + (-2C)$$

and each of these terms is positive, then the sum of the roots is positive, which means that they cannot both be negative, thus must both be positive.
Therefore, $h(x)$ has three distinct positive real roots, one of which is 1.

We have shown that if $f(x) = x^3 + Ax^2 + Bx + C$ has three distinct positive real roots, one of which is 1, then $h(x) = x^3 + (2B - A^2)x^2 + (B^2 - 2AC)x - C^2$ has three distinct positive real roots, one of which is 1.
The process of moving from $f(x)$ to $h(x)$ is exactly the process of moving from $f_{n-1}(x)$ to $f_n(x)$ for any positive integer $n \geq 2$.
Therefore, since $f_1(x)$ has three distinct positive real roots, one of which is 1, then $f_2(x)$ has this property, which in turn means that $f_3(x)$ has this property, and so on.
Continuing, this shows us that $f_{100}(x)$ has three distinct positive real roots.

3. (a) We note that $PB = PD = 53$ and $BC = 28$.
By the Pythagorean Theorem in $\triangle BCP$, we have

$$PC^2 = PB^2 - BC^2 = 53^2 - 28^2 = (53 + 28)(53 - 28) = 81 \cdot 25 = 9^2 \cdot 5^2$$

Since $PC > 0$, then $PC = 9 \cdot 5 = 45$.
Since $ABCD$ is a rectangle, then $AB = DC = PD + PC = 53 + 45 = 98$.

(b) Suppose that $BC = m$, an integer, and that $PD = x$.
Note that $PB = PD = x$.
Also, since $DC = AB = 101$, then $PC = DC - PD = 101 - x$.
Using the Pythagorean Theorem in $\triangle PBC$ gives the following equivalent equations:

$$\begin{aligned} PB^2 &= PC^2 + BC^2 \\ x^2 &= (101 - x)^2 + m^2 \\ x^2 &= x^2 - 202x + 101^2 + m^2 \\ 202x &= 101^2 + m^2 \\ x &= \frac{101^2 + m^2}{2 \cdot 101} \end{aligned}$$

For x to be an integer, we need the numerator to be divisible by 101.
Since 101^2 is divisible by 101, we need m^2 to be divisible by 101.
Since 101 is prime, we need m to be divisible by 101.
However, $m = BC < AB = 101$ and there are no positive multiples of 101 less than 101.
This means that m cannot be divisible by 101, which means that x, the length of PD, cannot be an integer.

(c) Suppose that $AB = n$, $BC = m$, and $PD = x$ for integers m, n and x with $m < n$ and $x < n$.
Here, we have $PB = x$, $PC = n - x$, and $BC = m$.
Using the Pythagorean Theorem in $\triangle PBC$ gives the following equivalent equations:

$$\begin{aligned} PB^2 &= PC^2 + BC^2 \\ x^2 &= (n-x)^2 + m^2 \\ x^2 &= x^2 - 2nx + n^2 + m^2 \\ 2nx &= n^2 + m^2 \\ x &= \frac{n^2+m^2}{2n} \end{aligned}$$

We need to determine all positive integers m with $1 \le m \le 100$ for which there are 7 positive integers n for which $x = \frac{n^2+m^2}{2n}$ is an integer.
We note that, since the demoninator of this fraction ($2n$) is even, then for x to be an integer, the numerator of this fraction ($n^2 + m^2$) is also even.
Since $n^2 + m^2$ must be even, then n and m are both even or both odd.
We look at two cases: m odd and m even.

<u>Case 1: m is odd</u>
Since m is odd, then n is odd.
Here, $x = \frac{n^2+m^2}{2n} = \frac{n}{2} + \frac{1}{2} \cdot \frac{m^2}{n}$.
Since $\frac{n}{2}$ is an odd integer divided by 2, then for x to be an integer, $\frac{1}{2} \cdot \frac{m^2}{n}$ must also equal an odd integer divided by 2, which means that $\frac{m^2}{n}$ must equal an odd integer.
Recalling that $n > m$, this means that we want to determine all odd integers m with $1 \le m \le 100$ for which exactly 7 integers n with $m < n \le m^2$ are divisors of m^2.
Note that every divisor n of m^2 with $m < n \le m^2$ will correspond to a divisor q of m^2 with $1 \le q < m$. This correspondence comes from the factor pair $qn = m^2$ and noting that $q = \frac{m^2}{n}$ from which $m < n \le m^2$ gives $\frac{m^2}{m^2} \le q < \frac{m^2}{m}$.
Additionally, m^2 has the divisor m for which we have not yet accounted.
This means that we want to determine all odd integers m with $1 \le m \le 100$ for which m^2 has exactly $2 \cdot 7 + 1 = 15$ positive divisors.
If m has three distinct prime divisors p_1, p_2, p_3, then m^2 is divisible by the product $p_1^2 p_2^2 p_3^2$.
This means that m^2 has at least $(2+1)(2+1)(2+1) = 27$ positive divisors, which is impossible.
Therefore, m can have at most two distinct prime divisors.
Suppose that $m = p_1^a$ for some prime p_1 and positive integer a.
Then $m^2 = p_1^{2a}$ and m^2 has $2a + 1$ positive divisors.
Since m^2 has 15 positive divisors, then $2a + 1 = 15$ which gives $a = 7$ and so $m = p_1^7$.
However, the smallest odd perfect seventh power is $3^7 > 100$, and so there are no m in this sub-case.
Suppose that $m = p_1^a p_2^b$ for some odd primes $p_1 \ne p_2$ and positive integers a and b.
Thus, $m^2 = p_1^{2a} p_2^{2b}$ and m^2 has $(2a + 1)(2b + 1)$ divisors.
Since $2a + 1 \ge 3$ and $2b + 1 \ge 3$ and $(2a + 1)(2b + 1) = 15$, then $2a + 1$ and $2b + 1$ must be 5 and 3 in some order, which means that a and b must be 2 and 1 in some order.

Thus, $m = p_1^2 p_2$ for some odd primes p_1 and p_2.
If $p_1 = 3$, then since $m = 9p_2$ and $1 \leq m \leq 100$, we can have $m = 45$ (from $p_2 = 5$) or $m = 63$ (from $p_2 = 7$) or $m = 99$ (from $p_2 = 11$).
If $p_1 = 5$, then since $m = 25p_2$ and $1 \leq m \leq 100$, we can have $m = 75$ (from $p_2 = 3$).
If $p_1 \geq 7$, there are no odd primes p_2 for which $m \leq 100$.
Therefore, in the case that m is odd, the possible values for m are $m = 45, 63, 75, 99$.

Case 2: m is even
Since m is even, then n is even.
Let $m = 2M$ and $n = 2N$ for some positive integers M and N.
Since $1 \leq 2M \leq 100$, then $0.5 \leq M \leq 50$ and so $1 \leq M \leq 50$.
Here, $x = \dfrac{4N^2 + 4M^2}{4N} = N + \dfrac{M^2}{N}$.
Since N is an integer, then for x to be an integer, $\dfrac{M^2}{N}$ must also equal an integer.
Since $n > m$, then $N > M$. This means that we want to determine all integers M with $1 \leq M \leq 50$ for which exactly 7 integers N with $M < N \leq M^2$ are divisors of M^2.
As above, this means that we want to determine all integers M with $1 \leq M \leq 50$ for which M^2 has exactly 15 positive divisors.
Again, as above, M must be of the form $p_1^2 p_2$ for some primes p_1 and p_2. This time, we do not have the restriction that M must be odd.
If $p_1 = 2$, then since $M = 4p_2$ and $1 \leq M \leq 50$, we can have $M = 12$ (from $p_2 = 3$) or $M = 20$ (from $p_2 = 5$) or $M = 28$ (from $p_2 = 7$) or $M = 44$ (from $p_2 = 11$).
If $p_1 = 3$, then since $M = 9p_2$ and $1 \leq M \leq 50$, we can have $M = 18$ (from $p_2 = 2$) or $M = 45$ (from $p_2 = 5$).
If $p_1 = 5$, then since $M = 25p_2$ and $1 \leq M \leq 50$, we can have $M = 50$ (from $p_2 = 2$).
Since $m = 2M$, the possible values of m in this case are $m = 24, 40, 56, 88, 36, 90, 100$.

Combining the two cases, the values of m are $m = 24, 36, 40, 45, 56, 63, 75, 88, 90, 99, 100$.