The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Canadian Senior Mathematics Contest

Wednesday, November 13, 2024
(in North America and South America)

Thursday, November 14, 2024
(outside of North America and South America)

Time: 2 hours

Calculating devices are allowed, provided that they do not have any of the following features: (i) internet access, (ii) the ability to communicate with other devices, (iii) information previously stored by students (such as formulas, programs, notes, etc.), (iv) a computer algebra system, (v) dynamic geometry software.

Do not open this booklet until instructed to do so.
There are two parts to this paper. The questions in each part are arranged roughly in order of increasing difficulty. The early problems in Part B are likely easier than the later problems in Part A.

PART A

1. This part consists of six questions, each worth 5 marks.
2. **Enter the answer in the appropriate box in the answer booklet.** For these questions, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded **only if relevant work** is shown in the space provided in the answer booklet.

PART B

1. This part consists of three questions, each worth 10 marks.
2. **Finished solutions must be written in the appropriate location in the answer booklet.** Rough work should be done separately. If you require extra pages for your finished solutions, they will be supplied by your supervising teacher. Insert these pages into your answer booklet. Write your name, school name, and question number on any inserted pages.
3. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

At the completion of the contest, insert your student information form inside your answer booklet.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on the website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some students may be shared with other mathematical organizations for other recognition opportunities.

Canadian Senior Mathematics Contest

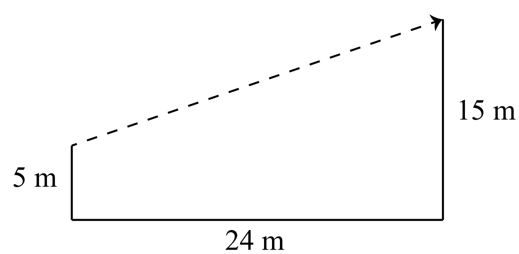
NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write solutions in the answer booklet provided.
3. Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.
4. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.
5. Diagrams are not drawn to scale. They are intended as aids only.
6. No student may write both the Canadian Senior Mathematics Contest and the Canadian Intermediate Mathematics Contest in the same year.

PART A

For each question in Part A, full marks will be given for a correct answer which is placed in the box. Part marks will be awarded only if relevant work is shown in the space provided in the answer booklet.

1. What integer is equal to $\sqrt{10^2 + 2 \cdot 10 \cdot 11 + 11^2}$?

2. Two vertical trees are 5 m and 15 m tall. The bases of the trees are 24 m apart along horizontal ground. A bird flies from the top of the shorter tree to the top of the taller tree along a straight path at a constant speed of 4 m/s. How long does it take the bird to complete the flight?

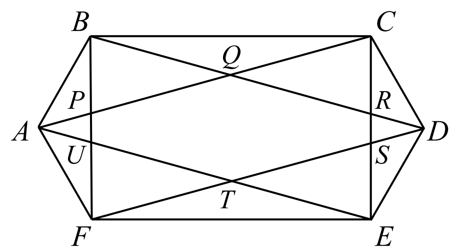


3. At the end of yesterday's soccer game between Team Why and Team Zed, Team Why had scored 3 goals and Team Zed had scored 2 goals. At half-time of the game, Team Why had scored y goals and Team Zed had scored z goals. If $y \geq 0$ and $z \geq 0$, how many possibilities are there for the ordered pair of integers (y, z)?

   (In soccer, each team's score is always a non-negative integer that never decreases as the game proceeds.)

4. For how many ordered quadruples (a, b, c, d) of positive integers with $d \leq 8$ is d equal to the product of a, b and c? (That is, for how many such ordered quadruples is $abc = d$?)

5. In the diagram, $ABCDEF$ is a hexagon with six equal interior angles (that is, $\angle ABC = \angle BCD = \angle CDE = \angle DEF = \angle EFA = \angle FAB$). Also, $BC = EF = 6$ and $AB = CD = DE = FA = 2$. Line segments AC, BD, CE, DF, EA, FB create a smaller hexagon $PQRSTU$, as shown. If the area of hexagon $PQRSTU$ is $\dfrac{\sqrt{n}}{t}$, where n and t are positive integers with t as small as possible, what is the ordered pair (n, t)?



6. A *Gleeson* list is an increasing list of distinct positive integers with a sum of 2024. For example, $70, 700, 1254$ is a Gleeson list of length 3 and $2, 4, 6, 10, 15, 987, 1000$ is a Gleeson list of length 7. Let M be the maximum possible length of a Gleeson list. How many Gleeson lists of length M are there?

PART B

For each question in Part B, your solution must be well-organized and contain words of explanation or justification. Marks are awarded for completeness, clarity, and style of presentation. A correct solution, poorly presented, will not earn full marks.

1. (a) Determine the integer x for which $2^{x+1} = 64$.

   (b) Determine the values of t and u that satisfy the following system of equations:

$$2^{t+u} = 2^8$$
$$5^{u-3} = 5^2$$

   (c) Determine the integers m and r for which $2^{2m+r}5^{3m-r} = 2^7 5^3$.

   (d) Show that there are no integers p and q for which $2^{p+q}5^{p-q} = 80$.

2. (a) The quadratic equation $x^2 - 2x - 1 = 0$ has solutions $x = r$ and $x = s$. Determine integers b and c for which the quadratic equation $x^2 + bx + c = 0$ has solutions $x = 2r + s$ and $x = r + 2s$.

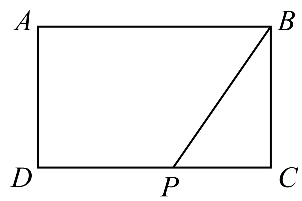
   (b) Suppose that m and p are real numbers for which the polynomial $f(x) = x^2 + mx + p$ has two distinct positive real roots. Prove that the polynomial $g(x) = x^2 - (m^2 - 2p)x + p^2$ has two distinct positive real roots.

   (c) Suppose that $A_1 = -6$, $B_1 = 10$ and $C_1 = -5$. For each positive integer $n \geq 2$, let

$$A_n = 2B_{n-1} - (A_{n-1})^2$$
$$B_n = (B_{n-1})^2 - 2A_{n-1}C_{n-1}$$
$$C_n = -(C_{n-1})^2$$

   Prove that the polynomial $f_{100}(x) = x^3 + A_{100}x^2 + B_{100}x + C_{100}$ has three distinct positive real roots.

3. In the diagram, $ABCD$ is a rectangle with $AB > BC$. Point P is on CD so that $PD = PB$.



(a) Suppose that $PD = 53$ and $BC = 28$. Determine the length of AB.

(b) Suppose that $AB = 101$. If the length of BC is an integer, prove that the length of PD cannot be an integer.

(c) Suppose that $BC = m$ for some positive integer m. Suppose further that, for this value of m, there are exactly 7 positive integers n so that when $AB = n$, the length of PD is an integer. Determine all possible values of m with $1 \leq m \leq 100$.