The **CENTRE for EDUCATION**
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2024 Canadian Intermediate Mathematics Contest

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(in North America and South America)

Thursday, November 14, 2024
(outside of North America and South America)

Solutions

Part A

1. *Solution 1*
   The container is in the shape of a cube, so no matter which face is sitting on the table, the base of the container is a square with side length 4 cm.
   The space occupied by the water in the container is a rectangular prism. The base of this rectangular prism is the base of the container and its height is the depth of the water.
   The volume of the water is $(4\,\text{cm}) \times (4\,\text{cm}) \times (2\,\text{cm}) = 32 \text{ cm}^3$.

   Solution 2
   The container is in the shape of a cube, so no matter which face is sitting on the table, the base of the container is a square with side length 4 cm.
   The depth of the water is half the height of the cube, so the volume of the water is half the volume of the cube.
   The volume of the cube is $(4\,\text{cm}) \times (4\,\text{cm}) \times (4\,\text{cm}) = 64 \text{ cm}^3$, so the volume of the water is $\dfrac{64}{2}\,\text{cm}^3 = 32\,\text{cm}^3$.

   ANSWER: 32

2. *Solution 1*
   Suppose the number of students surveyed is N.
   From the pie graph, the number of students who chose green is $0.35N$ and the number of students who chose blue is $0.45N$.
   All other students chose red, so the number of students who chose red is $N - 0.35N - 0.45N = 0.2N$.
   From the bar graph, the number of students who chose red is 8, and so we get $0.2N = 8$.
   Dividing both sides of this equation by 0.2 gives $N = \dfrac{8}{0.2} = 40$.

   Solution 2
   From the pie graph, 35% of the students chose green and 45% chose blue.
   The rest of the students chose red, so the percentage of the students that chose red must be $100\% - 35\% - 45\% = 20\%$.
   This means $\dfrac{20\%}{100\%} = \dfrac{1}{5}$ of the students chose red.
   From the bar graph, 8 students chose red.
   Therefore, the 8 students who chose red make up one-fifth of the students surveyed, so the total number of students surveyed is $8 \times 5 = 40$.

   ANSWER: 40

3. In an equilateral triangle, each angle measures $60°$, so $\angle CDB = 60°$.
   In a square, each angle measures $90°$, so $\angle BDE = 90°$.
   Using this, we have $\angle CDE = \angle CDB + \angle BDE = 60° + 90° = 150°$.
   Since $\triangle BCD$ is equilateral, $CD = BD$. Since $ABDE$ is a square, $ED = BD$. Therefore, $ED = BD = CD$.
   Since $CD = ED$, $\triangle CDE$ is isosceles, and so $\angle CED = \angle ECD$.
   The angles in a triangle have a sum of $180°$, so $180° = \angle CDE + \angle ECD + \angle CED$.
   Substituting $\angle CED = \angle ECD$ and $\angle CDE = 150°$ gives

   $$180° = 150° + \angle ECD + \angle ECD$$

which can be simplified to $30° = 2\angle ECD$, or $\angle ECD = 15°$.
Using again that $\triangle BCD$ is equilateral, we have $60° = \angle BCD = \angle ECB + \angle ECD$, and so

$$\angle ECB = 60° - \angle ECD = 60° - 15° = 45°$$

<div align="right">ANSWER: 45°</div>

4. Suppose that $a = 21$ and $b = 44$.
   Then $2a = 2 \times 21 = 42$, which is less than $b = 44$, so $2a < b$.
   As well, $\dfrac{a}{4} + \dfrac{b}{2} = \dfrac{21}{4} + \dfrac{44}{2} = 5.25 + 22 = 27.25$, which is greater than 27 and less than 28.
   We have shown that when $a = 21$, it is possible to choose b so that the conditions are satisfied.

   Now suppose $a \geq 22$. Then $2a \geq 2 \times 22$ or $2a \geq 44$.
   If we insist that $b > 2a$, then we will have $b > 44$.
   Since b is a positive integer and $b > 44$, it must be true that $b \geq 45$.
   Thus, if we assume $a \geq 22$, we must have that $b \geq 45$.
   Putting $a \geq 22$ and $b \geq 45$ together, we get

   $$\frac{a}{4} + \frac{b}{2} \geq \frac{22}{4} + \frac{45}{2} = 5.5 + 22.5 = 28$$

   and so $\dfrac{a}{4} + \dfrac{b}{2}$ is at least 28.

   It is given that $\dfrac{a}{4} + \dfrac{b}{2}$ must be less than 28, so we conclude that when $a \geq 22$, it is not possible to choose b so that all conditions are satisfied.
   Therefore, the greatest possible value of a is 21.

   Note: To see how one might arrive at the number 21, observe that $\dfrac{a}{4} + \dfrac{b}{2} = \dfrac{a}{4} + \dfrac{2b}{4} = \dfrac{a + 2b}{4}$,
   so we must have $\dfrac{a + 2b}{4} < 28$.
   Multiplying by 4, we get $a + 2b < 4 \times 28$ or $a + 2b < 112$.
   In the inequality $2a < b$, we can multiply by 2 to get $4a < 2b$.
   Add a to both sides of this inequality to get $a + 4a < a + 2b$ or $5a < a + 2b$.
   Combining $5a < a + 2b$ with $a + 2b < 112$ gives $5a < 112$ or $a < \dfrac{112}{5} = 22.4$.
   The answer of 21 can now be found by starting at 22 and working backwards.

<div align="right">ANSWER: 21</div>

5. We will refer to the seven columns from left to right as C1, C2, and so on to C7.
   In every row, the integer in C1 is 5.
   In the first row, the integer in C2 is 5. In the second row, the integer in C2 is 6. In the third row, the integer in C2 is 7. As we follow C2 downward, every integer is one more than the previous integer, so we conclude that in the nth row, the integer in C2 is $n + 4$.

   The first two integers in the nth row are 5 and $n + 4$, so we can compute the rest of the nth row using the third bullet point in the problem statement.

   - The third integer in the nth row is $5 + (n + 4) = n + 9$.
   - The fourth integer in the nth row is $(n + 4) + (n + 9) = 2n + 13$.

- The fifth integer in the nth row is $(n+9) + (2n+13) = 3n+22$.

- The sixth integer in the nth row is $(2n+13) + (3n+22) = 5n+35$.

- The seventh integer in the nth row is $(3n+22) + (5n+35) = 8n+57$.

To summarize, the integers in the nth row, from left to right, are

$$5, \quad n+4, \quad n+9, \quad 2n+13, \quad 3n+22, \quad 5n+35, \quad 8n+57$$

We are looking for two-digit integers that appear in exactly five of the cells. Since C1 contains no two-digit integers, it can be ignored.

In C2 through C7, the integers increase moving down the column, so none of these columns contains the same integer multiple times.

We conclude that we are looking for two-digit integers appearing in exactly five of the columns.

The integers in C2 are $n+4$ where n ranges from 1 through 95, or the integers 5 through 99.
The integers in C3 are $n+9$ where n ranges from 1 through 95, or the integers 10 through 104.
Thus, C2 and C3 each contain *every* two-digit integer, so we are really looking for two-digit integers that are in exactly 3 of C4, C5, C6, and C7.

The integers in C7 are those of the form $8n+57$, the first few of which are $8(1)+57 = 65$, $8(2)+57 = 73$, $8(3)+57 = 81$, $8(4)+57 = 89$, and $8(5)+57 = 97$.
All other integers in C7 have at least three digits, so the only two-digit integers in C7 are 65, 73, 81, 89, and 97.
We will consider two cases: Integers not in C7 but in C4, C5, and C6, and integers in C7 and in exactly two of C4, C5, and C6.

<u>Two-digit integers that are in C4, C5, and C6, but not in C7</u>: Suppose a two-digit integer x is in C4, C5, and C6, but not in C7.
The integers in C6 are of the form $5n+35$, and so are multiples of 5 that are greater than 35.
The integers in C4 are of the form $2n+13$ which are odd integers that are greater than 13.
We conclude that x is a two-digit odd multiple of 5 that is greater than 35, so it must be one of 45, 55, 65, 75, 85, or 95.
We need to check which of these values of x are in C5.

In general, to determine whether an integer x is in a given column, we can set x equal to the expression for that column and solve for n. If the solution n is a positive integer from 1 through 95, then x is in the nth row (in the column of interest). Otherwise, x is not in that column.
For example, 55 is in C5 because if we set $55 = 3n+22$, we can solve to get $n = 11$, so 55 is in the 11th row. On the other hand, 45 is not in C5 because if we set $45 = 3n+22$ and solve we get $n = \dfrac{23}{3}$, which is not an integer (if 45 were in C5, it would have to be in row $\frac{23}{3}$, which does not make sense).

Using the idea in the previous paragraph, we will solve $x = 3n+22$ for n for each x in the list 45, 55, 65, 75, 85, 95. The x-values that are in C4 are those that lead to an integer solution n.
The x-values lead to n values of $\dfrac{23}{3}$, 11, $\dfrac{43}{3}$, $\dfrac{53}{3}$, 21, and $\dfrac{73}{3}$, respectively.
Only 55 and 85 give an integer solution n, so 55 and 85 are the only two-digit integers in C4, C5, and C6.
Neither 55 nor 85 is in C7, so they are the only two-digit integers that are in C4, C5, and C6, but not in C7.

<u>Two-digit integers that are in C7 and exactly two of C4, C5, and C6</u>: In the table below, we check each of the two-digit integers in C7 to see in which of C4, C5, and C6 they also appear.

x	solution to $x = 2n + 13$	in C4?	solution to $x = 3n + 22$	in C5?	solution to $x = 5n + 35$	in C6?	# of columns
65	$n = 26$	YES	$n = \frac{43}{3}$	NO	$n = 6$	YES	2
73	$n = 30$	YES	$n = 17$	YES	$n = \frac{38}{5}$	NO	2
81	$n = 34$	YES	$n = \frac{59}{3}$	NO	$n = \frac{46}{5}$	NO	1
89	$n = 38$	YES	$n = \frac{67}{3}$	NO	$n = \frac{54}{5}$	NO	1
97	$n = 42$	YES	$n = 25$	YES	$n = \frac{62}{5}$	NO	2

From the table, we see that 65, 73, and 97 are the only two-digit integers that are in C7 and exactly two of C4, C5, and C6.

Hence, of the two-digit integers in C7, exactly three are in exactly five cells in the grid.

Combining the two cases, there are five two-digit integers that are in exactly five cells. They are 55, 65, 73, 85, and 97.

ANSWER: 5

6. *Solution 1*

We will compute the probability that the condition fails, then subtract the result from 1.

The given condition is that at least one diameter has a multiple of 3 at one end and a multiple of 2 at the opposite end.

For this condition to fail, there must be no diameter with this property.

There are only two available multiples of 3, and they are 3 and 6.

Therefore, to ensure the condition fails, we need to ensure that the diameters containing 3 and 6 do not have an even integer at the other end.

Consider the diameter with 3 at one end. At the other end must be an odd integer, but we cannot use 3 twice, so we must place 1, 5, or 7 at the other end.

Similarly, the diameter with 6 at one end must have an odd integer at its other end, but we cannot use 3 since this would mean an even integer is at the other end of the diameter with 3.

Therefore, the condition fails exactly when 3 and 6 are on different diameters and the two diameters with 3 and 6 each have one of 1, 5, or 7 at their other end. We will now compute the probability that this happens.

There are 8 possibilities for the position of the integer 1.

Once 1 is is placed, there are 7 possibilities for the position of the integer 2.

Continuing in this way, once 1 and 2 are placed, there are 6 possibilities for the position of the integer 3, then 5 possibilities for the integer 4, and so on until we get to the integer 8, which will have only one possibility for where it can be placed.

Because of this, there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ possible ways to place the integers.

Next, we will count the ways to place the integers so that 3 and 6 both share a diameter with one of 1, 5, or 7.

There are 8 ways to place the integer 3.

After the integer 3 is placed, there are 3 choices for the integer at the other end of the diameter containing 3, for a total of $8 \times 3 = 24$ ways to place 3 and the other integer on its diameter.

Once these integers are placed, there are 6 ways to place the 6.

One of the integers from 1, 5, and 7 has already been used, so there are 2 choices for the integer at the other end of the diameter containing 6.

Thus, there are $8 \times 3 \times 6 \times 2$ ways to select and place the integers on the diameters containing 3 and 6.

The remaining four integers, whatever they happen to be, can now be placed arbitrarily and the condition will still be guaranteed to fail.

By reasoning similar to earlier, there are $4 \times 3 \times 2 \times 1$ ways to place the remaining four integers. The probability that the condition fails is

$$\frac{8 \times 3 \times 6 \times 2 \times 4 \times 3 \times 2}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2} = \frac{3 \times 2}{7 \times 5} = \frac{6}{35}$$

and so the probability that the condition is satisfied is $1 - \dfrac{6}{35} = \dfrac{29}{35}$.

Solution 2

As explained in Solution 1, there are $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40\,320$ possible ways to arrange the integers. We will count the number of ways to arrange the integers so that the condition is satisfied, then divide this count by $40\,320$.

The only multiples of 3 are 3 and 6, so for the condition to be satisfied, one of 3 and 6 needs to share a diameter with an even number.

We will consider four cases.

<u>3 and 6 share a diameter</u>: In this case, 3 shares a diameter with the even integer 6, so the condition is satisfied regardless of the placement of the other integers.

There are 4 choices for which diameter contains 3 and 6 and 2 ways to place 3 and 6, for a total of 8 ways to place 3 and 6 on that diameter.

There are $6 \times 5 \times 4 \times 3 \times 2 \times 1$ ways to place the remaining integers.

Therefore, there are $8 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5760$ ways to place the integers in this case.

In the remaining three cases, 3 and 6 are on different diameters.

<u>3 and 6 are both on a diameter with an even number</u>: There are 8 choices for where to place 3, and then 6 ways to place the 6 (since it cannot go on the same diameter as 3 in this case).

The remaining even integers are 2, 4, and 8. There are 3 choices for which even integer to place on the diameter with 3, then two choices for which even integer to place on the diameter with 6.

After these four integers are placed, there are four remaining integers that can be placed in any of $4 \times 3 \times 2 \times 1$ ways.

There are $8 \times 6 \times 3 \times 2 \times 4 \times 3 \times 2 \times 1 = 6912$ ways to place the integers in this case.

<u>3 is on a diameter with an even integer and 6 is on a diameter with an odd integer</u>: As with the previous case, there are 8×6 ways to place 3 and 6.

The remaining even integers are 2, 4, and 8, and the remaining odd integers are 1, 5, and 7. There are 3 choices for the integer on the the diameter with 3 and 3 choices for the integer on the diameter with 6.

After these integers are placed, there are $4 \times 3 \times 2 \times 1$ ways to place the remaining integers.

There are $8 \times 6 \times 3 \times 3 \times 4 \times 3 \times 2 \times 1 = 10\,368$ ways to place the integers in this case.

<u>3 is on a diameter with an odd integer and 6 is on a diameter with an even integer</u>: This case has the same count as the previous case.

The probability is

$$\frac{5760 + 6912 + 2 \times 10\,368}{40\,320} = \frac{33\,408}{40\,320} = \frac{29}{35}$$

<div align="right">ANSWER: $\frac{29}{35}$</div>

Part B

1. Throughout this solution, when we refer to the "difference" between two numbers, we mean the larger of the two numbers minus the smaller. For example, the difference between 3 and 5 is 2 since $5 - 3 = 2$.

    (a) The coordinates of B are $(6, 3)$. This is because AB is horizontal, so A and B have the same y-coordinate, and BC is vertical, so B and C have the same x-coordinate.
    The length of AB is the difference between the x-coordinates of A and B, or $6 - 2 = 4$.
    The length of BC is the difference between the y-coordinates of B and C, or $7 - 3 = 4$.
    The area of $\triangle ABC$ is $\frac{1}{2} \times AB \times BC = \frac{1}{2} \times 4 \times 4 = 8$.

    (b) The height from X to YZ is equal to the difference between the y-coordinate of X and the common y-coordinate of X and Z, or $10 - 4 = 6$.
    The length of YZ is equal to the difference between the x-coordinates of Y and Z, or $a - 3$.
    The area of $\triangle XYZ$ is $\frac{1}{2} \times YZ \times 6 = \frac{1}{2} \times (a - 3) \times 6 = 3(a - 3)$.
    This area is given to be 24, so we have $24 = 3(a-3)$, or $8 = a - 3$, which can be rearranged to get $a = 8 + 3 = 11$.

    (c) Quadrilateral $PQRS$ can be split into the triangles $\triangle RQS$ and $\triangle PQS$. These triangles have a common base of QS and the sum of their areas is the area of $PQRS$.
    The height of $\triangle RQS$ is the difference between the y-coordinate of R and the common y-coordinate of Q and S, or $c + 2 - c = 2$.
    The height of $\triangle PQS$ is the difference between the y-coordinate of P and the common y-coordinate of Q and S, or $c - 4$.
    The length of QS is the difference between the x-coordinates of Q and S, or $2c - c = c$.
    The area of $\triangle RQS$ is $\frac{1}{2} \times QS \times 2 = \frac{1}{2} \times c \times 2 = c$.
    The area of $\triangle PQS$ is $\frac{1}{2} \times QS \times (c - 4) = \frac{1}{2}c(c - 4)$.
    The area of quadrilateral $PQRS$ is the sum of the areas of $\triangle RQS$ and $\triangle PQS$, which is

    $$c + \tfrac{1}{2}c(c - 4) = c + \tfrac{1}{2}c^2 - \tfrac{4}{2}c = \tfrac{1}{2}c^2 - c = \tfrac{1}{2}(c^2 - 2c) = \tfrac{1}{2}c(c - 2)$$

    It is given that the area of $PQRS$ is 180, so we have $180 = \frac{1}{2}c(c - 2)$ or $360 = c(c - 2)$.
    Since c, and hence, $c - 2$ are integers, we are looking for two integers that differ by 2 and have a product of 360.
    Two such integers are $c = 20$ and $c - 2 = 18$, so the answer to the question is $c = 20$.
    To see how to arrive at $c = 20$, one could list the factor pairs of 360 to eventually find $18 \times 20 = 360$. Alternatively, the equation $360 = c(c - 2)$ can be solved directly by rearranging to $c^2 - 2c - 360 = 0$ and either factoring or using the quadratic formula.

2. (a) The time that it takes to run 20 km at a speed of 12 km/h is

    $$\frac{20\,\text{km}}{12\,\text{km/h}} = \frac{20}{12}\,\text{h} = \frac{5}{3}\,\text{h}$$

    The time it takes to run 10 km at a speed of 10 km/h is 1 h.
    The total time it took Beryl to run 30 km is $\dfrac{5}{3}\,\text{h} + 1\,\text{h} = \dfrac{8}{3}\,\text{h}$.

    (b) The time that it took Carol to walk the first x km was $\dfrac{x\,\text{km}}{6\,\text{km/h}} = \dfrac{x}{6}\,\text{h}$.

    The time that it took Carol to walk the remaining $(10-x)$ km was $\dfrac{(10-x)\,\text{km}}{4\,\text{km/h}} = \dfrac{10-x}{4}\,\text{h}$.

In total, the walk took 2 hours and 18 minutes, or $2\frac{18}{60} = \frac{138}{60}$ h.

We now have the equation $\frac{138}{60} = \frac{x}{6} + \frac{10-x}{4}$.

Multiplying through by 60 leads to or $138 = 10x + 15(10-x)$ or $138 = 10x + 150 - 15x$.

Rearranging this equation gives $12 = 5x$ or $x = \frac{12}{5}$.

(c) The time that Daryl was riding at 24 km/h was $\frac{a\,\text{km}}{24\,\text{km/h}} = \frac{a}{24}$ h.

The time that Daryl was riding at 16 km/h was $\frac{b\,\text{km}}{16\,\text{km/h}} = \frac{b}{16}$ h.

The total time that Daryl was riding was 3 h, so $\frac{a}{24} + \frac{b}{16} = 3$.

Multiplying through by 48 gives $2a + 3b = 144$. We are looking for positive integer solutions (a, b) to this equation.

Since $2a$ is even and 144 is even, the integer $3b$ must also be even.

In order for $3b$ to be even, b must be even since 3 is odd. Therefore, there is some integer n such that $b = 2n$.

Substituting $b = 3n$ into $2a + 3b = 144$, we get $2a + 6n = 144$ or $a + 3n = 72$.

Since $3n$ and 72 are both multiples of 3, a must be a multiple of 3. Therefore, there is some integer m such that $a = 3m$.

Substituting $a = 3m$ into $a + 3n = 72$ gives $3m + 3n = 72$ or $m + n = 24$. The pairs (m, n) of positive integers that satisfy $m + n = 24$ are $(1, 23)$, $(2, 22)$, $(3, 21)$, and so on up to $(23, 1)$ for a total of 23 pairs.

Since $a = 3m$ and $b = 2n$, every pair (m, n) satisfying $m + n = 24$ corresponds to exactly one pair (a, b) satisfying $2a + 3b = 144$. For example, $(m, n) = (1, 23)$ corresponds to $(a, b) = (3, 46)$, and indeed $2(3) + 3(46) = 144$.

There are 23 pairs (m, n) satisfying $m + n = 24$, so there are 23 pairs (a, b) satisfying $2a + 3b = 144$.

(d) By similar reasoning to that which was used at the beginning of the solution to part (c), Errol spent $\frac{r}{12}$ h running, $\frac{j}{8}$ h jogging, and $\frac{w}{4}$ h walking.

Thus, $\frac{r}{12} + \frac{j}{8} + \frac{w}{4} = 5$, which can be multiplied through by 24 to get $2r + 3j + 6w = 120$.

We are looking for positive integer solutions to this equation.

Since $3j$, $6w$, and 120 are all multiples of 3, $2r$ must also be a multiple of 3, which means r itself is a multiple of 3 (by similar reasoning to that which was used in the solution to part (c)).

There is some positive integer k such that $r = 3k$, from which we get $6k + 3j + 6w = 120$. Dividing through by 3 gives $2k + j + 2w = 40$.

Since $2k$, $2w$, and 40 are all even, j must also be even, so there is some positive integer m such that $j = 2m$.

Substituting this into $2k + j + 2w = 40$ gives $2k + 2m + 2w = 40$, or $k + m + w = 20$.

Since every value of k corresponds to exactly one value of r and every value of m corresponds to exactly one value of m, we can count the triples (r, j, w) by counting the triples (k, m, w) of positive integers that satisfy $k + m + w = 20$.

Suppose $k = 1$. Then $1 + m + w = 20$, so $m + w = 19$.

There are 18 positive integer solutions (m, w) to this equation: $(1, 18)$, $(2, 17)$, $(3, 16)$, and so on up to $(18, 1)$, so there are a total of 18 solutions with $k = 1$.

If $k = 2$, then $m + w = 18$. There are 17 positive integer solutions (m, w) to $m + w = 18$:

$(1, 17)$, $(2, 16)$, and so on up to $(17, 1)$, so there are a total of 17 solutions with $k = 2$. Continuing in this way, there are 16 solutions when $k = 3$, there are 15 solutions when $k = 4$, and so on, up to there being 1 solution when $k = 18$.

When $m = w = 1$, we have $k + 1 + 1 = 20$, so $k = 18$. Thus, when m and w are minimized, k is 18, so 18 is the largest that k can be.

Therefore, the number of positive integer solutions is

$$1 + 2 + 3 + \cdots + 18 = 171$$

3. (a) The 3-sign sum is $8 + 9 - 10 + 11 + 12 - 13 + 14 + 15 = 46$.

    (b) *Solution 1*

    Consider a slice of the list 1, 2, 3, ..., $n - 1$, n. If the slice does not contain the integer n, then it is also a slice of the shorter list 1, 2, 3, ..., $n - 1$. Moreover, every slice of 1, 2, 3, ..., $n - 1$ is a slice of 1, 2, 3, ..., $n - 1$, n.

    From the previous paragraph, we deduce that the slices of 1, 2, 3, ..., $n - 1$, n fall into two categories: The slices of 1, 2, 3, ..., $n - 1$, and the slices that contain the integer n. The quantity G_n can be computed as the sum of the 3-sign sums of all slices of 1, 2, 3, ..., $n - 1$ plus the 3-sign sums of all slices that contain n.

    In other words, G_n is equal to G_{n-1} plus the 3-sign sums of all slices of 1, 2, 3, ..., $n - 1$, n that contain n.

    Using this, we get that the quantity $G_n - G_{n-1}$ is exactly equal to the sum of the 3-sign sums of the slices 1, 2, 3, ..., n that contain n.

    For example, $G_6 - G_5$ is equal to

    $$
    \begin{aligned}
    &1 + 2 - 3 + 4 + 5 - 6 \\
    &+ 2 + 3 - 4 + 5 + 6 \\
    &+ 3 + 4 - 5 + 6 \\
    &+ 4 + 5 - 6 \\
    &+ 5 + 6 \\
    &+ 6 = 43
    \end{aligned}
    $$

    and $G_5 - G_4$ is equal to

    $$
    \begin{aligned}
    &1 + 2 - 3 + 4 + 5 \\
    &+ 2 + 3 - 4 + 5 \\
    &+ 3 + 4 - 5 \\
    &+ 4 + 5 \\
    &+ 5 = 31
    \end{aligned}
    $$

    Now observe that $G_6 - 2G_5 + G_4 = (G_6 - G_5) - (G_5 - G_4) = 43 - 31 = 12$.

    The way the sums are expressed above, it might become apparent that there is simpler way to see that this difference equals 12 by noticing that a lot of cancellation occurs. For instance, the first "lines" in the two sums above are almost identical with the exception of a -6. The second "lines" are also identical with the exception of a $+6$.

    To generalize this, we will introduce the notation $\langle u, v \rangle$ for the 3-sign sum of the list of integers from u to v, inclusive. For example, $\langle 3, 8 \rangle = 3 + 4 - 5 + 6 + 7 - 8$. Note that $\langle u, u \rangle$ is just the integer u (the 3-sign sum of a "list" consisting of exactly one integer). With this notation and from the expression above, we can see that

    $$G_6 - G_5 = \langle 1, 6 \rangle + \langle 2, 6 \rangle + \langle 3, 6 \rangle + \langle 4, 6 \rangle + \langle 5, 6 \rangle + \langle 6, 6 \rangle$$

and more generally, we have

$$G_n - G_{n-1} = \langle 1, n \rangle + \langle 2, n \rangle + \langle 3, n \rangle + \cdots + \langle n-1, n \rangle + \langle n, n \rangle$$

Now consider the 3-sign sums $\langle 1, n \rangle$ and $\langle 1, n-1 \rangle$.

These 3-sign sums are identical except for the appearance of $\pm n$ at the end of $\langle 1, n \rangle$.

Thus, $\langle 1, n \rangle - \langle 1, n-1 \rangle$ is either n or $-n$.

More generally, if $k \leq n-1$, then the 3-sign sums $\langle k, n \rangle$ and $\langle k, n-1 \rangle$ are identical except for the $\pm n$ at the end of $\langle k, n \rangle$, so $\langle k, n \rangle - \langle k, n-1 \rangle$ is $\pm n$, and the sign depends on the number of terms in $\langle k, n \rangle$.

Specifically, $\langle k, n \rangle - \langle k, n-1 \rangle$ is $-n$ if the number of terms in $\langle k, n \rangle$ is a multiple of 3, and it is $+n$ otherwise.

We can use this to find the exact value of the expression

$$G_{3n} - 2G_{3n-1} + G_{3n-2} = (G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})$$

Using our notation, we have

$$G_{3n} - G_{3n-1} = \langle 1, 3n \rangle + \langle 2, 3n \rangle + \langle 3, 3n \rangle + \cdots + \langle 3n-1, 3n \rangle + \langle 3n, 3n \rangle$$

and

$$G_{3n-1} - G_{3n-2} = \langle 1, 3n-1 \rangle + \langle 2, 3n-1 \rangle + \langle 3, 3n-1 \rangle + \cdots + \langle 3n-2, 3n-1 \rangle + \langle 3n-1, 3n-1 \rangle$$

and so we can arrange the difference $(G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})$ to be

$$(\langle 1, 3n \rangle - \langle 1, 3n-1 \rangle) + (\langle 2, 3n \rangle - \langle 2, 3n-1 \rangle) + \cdots + (\langle 3n-1, 3n \rangle - \langle 3n-1, 3n-1 \rangle) + \langle 3n, 3n \rangle$$

The quantity $\langle 1, 3n \rangle$ has $3n$ terms, so $\langle 1, 3n \rangle - \langle 1, 3n-1 \rangle = -3n$.

The quantity $\langle 2, 3n \rangle$ has $3n-1$ terms (not a multiple of 3), so $\langle 2, 3n \rangle - \langle 2, 3n-1 \rangle = 3n$.

The quantity $\langle 3, 3n \rangle$ has $3n-2$ terms (not a multiple of 3), so $\langle 3, 3n \rangle - \langle 3, 3n-1 \rangle = 3n$.

Continuing, we have that $\langle 4, 3n \rangle$ has $3n-3$ terms, which is a multiple of 3, and so $\langle 4, 3n \rangle - \langle 4, 3n-1 \rangle = -3n$.

This pattern continues and we get that $(G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})$ is equal to a sum of the form

$$(-3n) + (3n) + (3n) + (-3n) + (3n) + (3n) + \cdots + (-3n) + (3n) + (3n)$$

where there are exactly $3n$ terms, n of which equal $-3n$ and $2n$ of which are $3n$.

Therefore, we have

$$\begin{aligned} \frac{G_{3n} - 2G_{3n-1} + G_{3n-2}}{3} &= \frac{(G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})}{3} \\ &= \frac{n(-3n) + 2n(3n)}{3} \\ &= \frac{-3n^2 + 6n^2}{3} \\ &= \frac{3n^2}{3} \\ &= n^2 \end{aligned}$$

which is a perfect square.

Solution 2

With notation and reasoning as in Solution 1, we have that

$$G_n - G_{n-1} = \langle 1, n \rangle + \langle 2, n \rangle + \langle 3, n \rangle + \cdots + \langle n - 1, n \rangle + \langle n, n \rangle$$

We will find an explicit form for this sum by examining the total contribution of each integer across the sum of 3-sign sums.

For example, $G_9 - G_8 = \langle 1, 9 \rangle + \langle 2, 9 \rangle + \langle 3, 9 \rangle + \cdots + \langle 9, 9 \rangle$ which can be expressed as

$$
\begin{aligned}
1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 \\
+ 2 + 3 - 4 + 5 + 6 - 7 + 8 + 9 \\
+ 3 + 4 - 5 + 6 + 7 - 8 + 9 \\
+ 4 + 5 - 6 + 7 + 8 - 9 \\
+ 5 + 6 - 7 + 8 + 9 \\
+ 6 + 7 - 8 + 9 \\
+ 7 + 8 - 9 \\
+ 8 + 9 \\
+ 9
\end{aligned}
$$

We can compute this sum as follows: The integer 1 occurs exactly once with a + sign, so the total of all 1s in the sum is 1. The integer 2 occurs twice with a + sign, so the total contribution of 2s to the sum is $2(2) = 4$. The integer 3 occurs twice with a + sign and once with a − sign, so the total contribution of the 3s is $3 + 3 - 3 = (2 - 1)3 = 3$. The integer 4 occurs three times with a + sign and once with a − sign, so its total contribution of the 4s is $(3 - 1)4 = 8$. Continuing in this way, the sum $G_9 - G_8$ is

$$(1)1 + (2)2 + (2 - 1)3 + (3 - 1)4 + (4 - 1)5 + (4 - 2)6 + (5 - 2)7 + (6 - 2)8 + (6 - 3)9 = 123$$

In general, consider the integer 1 in $\langle 1, n \rangle + \langle 2, n \rangle + \langle 3, n \rangle + \cdots + \langle n - 1, n \rangle + \langle n, n \rangle$.

It occurs in $\langle 1, n \rangle$ as $+1$ but does not occur in $\langle 2, n \rangle$, $\langle 3, n \rangle$, or any of the other 3-sign sums. Thus, the total contribution of all 1s is $1(1) = 1$.

The integer 2 occurs in $\langle 1, n \rangle$ and $\langle 2, n \rangle$, each as $+2$, and does not occur in any other 3-sign sums. Thus, the total contribution of all 2s is $2(2) = 4$.

The integer 3 occurs in $\langle 1, n \rangle$ as -3 and it occurs in $\langle 2, n \rangle$ and $\langle 3, n \rangle$ as $+3$, so the total contribution of all 3s is $(2 - 1)(3) = 3$.

Continuing in this way, 4 occurs in $\langle 1, n \rangle$ with a + sign, $\langle 2, n \rangle$ with a − sign, and $\langle 3, n \rangle$ and $\langle 4, n \rangle$ with a + sign. Therefore, the total contribution of all 4s is $(3 - 1)(4) = 8$.

In general, if k is an integer with $1 \leq k \leq n$, then k occurs in the sum $G_n - G_{n-1}$ a total of k times. The number of these occurrences that have a − sign depends on the particular value of k.

Suppose k is an integer satisfying $1 \leq k \leq n$.

The integer k occurs in $\langle k, n \rangle$ with a + sign. It occurs in $\langle k - 1, n \rangle$ with a + sign, and it occurs in $\langle k - 2, n \rangle$ with a − sign.

Continuing, it occurs in $\langle k - 3, n \rangle$ and $\langle k - 4, n \rangle$ with a + sign, and it occurs in $\langle k - 5, n \rangle$ with a − sign.

This pattern continues with two + signs followed by a − sign, then two more + signs and a − sign, and so on. Note that k does not occur at all in $\langle m, n \rangle$ when $m > k$.

If k is a multiple of 3 ($k = 3m$ for some integer m), then it will occur with m times with

a $-$ sign and $2m$ times with a $+$ sign. In this situation, the total contribution of k is

$$(2m - m)k = mk = \frac{k}{3} \times k = \frac{k^2}{3}$$

If k is 1 more than a multiple of 3 ($k = 3m + 1$ for some integer m), then k occurs m times with a $-$ sign and $2m + 1$ times with a $+$ sign. In this situation, the total contribution of k is

$$(2m + 1 - m)k = (m + 1)k = \left(\frac{k - 1}{3} + 1\right)k = \frac{k(k + 2)}{3}$$

If k is 2 more than a multiple of 3 ($k = 3m + 2$ for some integer m), then k occurs m times with a $-$ sign and $2m + 2$ times with a $+$ sign. In this situation, the total contribution of k is

$$(2m + 2 - m)k = (m + 2)k = \left(\frac{k - 2}{3} + 2\right)k = \frac{k(k + 4)}{3}$$

We now have a general expression for the contribution of k in the sum $G_n - G_{n-1}$ where $1 \le k \le n$.

Observe that the contribution never depends on n, which implies that if $k \le n - 1$, the contribution of k is exactly the same in $G_n - G_{n-1}$ and $G_{n-1} - G_{n-2}$.

Thus, in the quantity $G_{3n} - 2G_{3n-1} + G_{3n-2} = (G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})$, the contributions of all integers $k \le n - 1$ cancel out, so $(G_{3n} - G_{3n-1}) - (G_{3n-1} - G_{3n-2})$ must be equal to the contribution of $3n$ in $G_{3n} - G_{3n-1}$.

From the calculations above, since $3n$ is a multiple of 3, this contribution is

$$\frac{(3n)^2}{3} = 3n^2$$

from which it follows that

$$\frac{G_{3n} - 2G_{3n-1} + G_{3n-2}}{3} = \frac{3n^2}{3} = n^2$$

which is a perfect square.

(c) *Solution 1*

Using the work from part (b), $G_{2025} - G_{2024}$ is equal to the sum

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 2024, 2025 \rangle + \langle 2025, 2025 \rangle$$

The solution that follows will compute the remainder when this sum is divided by 27 without actually computing the sum.

First, let's focus on the sum of the first nine terms

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$$

For a fixed positive integer a, consider the following sum, paying careful attention to the location of the $-$ signs:

$$\begin{aligned} &a + (a + 1) - (a + 2) + (a + 3) + (a + 4) - (a + 5) + (a + 6) + (a + 7) - (a + 8) \\ &+a - (a + 1) + (a + 2) + (a + 3) - (a + 4) + (a + 5) + (a + 6) - (a + 7) + (a + 8) \quad (*) \\ &-a + (a + 1) + (a + 2) - (a + 3) + (a + 4) + (a + 5) - (a + 6) + (a + 7) + (a + 8) \end{aligned}$$

It can be checked that this sum is equal to $9a + 36$. Observe that 3 times this sum is $3(9a + 36) = 27a + 108 = 27(a + 4)$, which is a multiple of 27.

The 3-sign sums $\langle 1, 2025 \rangle$, $\langle 4, 2025 \rangle$, and $\langle 7, 2025 \rangle$ each contain the sum

$$10 + 11 - 12 + 13 + 14 - 15 + 16 + 17 - 18$$

The 3-sign sums $\langle 2, 2025 \rangle$, $\langle 5, 2025 \rangle$, and $\langle 8, 2025 \rangle$ each contain the sum

$$-10 + 11 + 12 - 13 + 14 + 15 - 16 + 17 + 18$$

The 3-sign sums $\langle 3, 2025 \rangle$, $\langle 6, 2025 \rangle$, and $\langle 9, 2025 \rangle$ each contain the sum

$$10 - 11 + 12 + 13 - 14 + 15 + 16 - 17 + 18$$

The three sums above can be totalled using the formula for (*) with $a = 10$. Since each sum occurs exactly three times, we conclude that in

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$$

if we look at the total of all occurrences of the integers from 10 through 18, with the appropriate signs, we will get $27(10 + 4)$, which is a multiple of 27.

By similar reasoning with $a = 19$, the sum of all occurrences of the integers 19 through 27, with the appropriate signs, is $27(19 + 4)$, which is a multiple of 27.

Continuing in this way, we can total all occurrences of 28 through 36 to get a multiple of 27, all occurrences of 37 through 45 to get a multiple of 27, and so on, up to all occurrences of 2017 through 2025 to get a multiple of 27. This works out neatly because $2025 = 9 \times 225$ is a multiple of 9.

In conclusion, in the sum

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$$

the sum of all occurrences of the integers 10 through 2025 inclusive is the sum of multiples of 27, and so is itself a multiple of 27.

Thus, $\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$ is equal to a multiple of 27 plus the following sum:

$$
\begin{aligned}
1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 & \\
+ 2 + 3 - 4 + 5 + 6 - 7 + 8 + 9 & \\
+ 3 + 4 - 5 + 6 + 7 - 8 + 9 & \\
+ 4 + 5 - 6 + 7 + 8 - 9 & \\
+ 5 + 6 - 7 + 8 + 9 & \qquad (**)\\
+ 6 + 7 - 8 + 9 & \\
+ 7 + 8 - 9 & \\
+ 8 + 9 & \\
+ 9 &
\end{aligned}
$$

which is equal to 123 (this was discussed in Solution 2 to part (b)).

Therefore, $\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$ is equal to 123 plus some multiple of 27.

By nearly identical reasoning, the sum of the next nine terms,

$$\langle 10, 2025 \rangle + \langle 11, 2025 \rangle + \langle 12, 2025 \rangle + \cdots + \langle 18, 2025 \rangle$$

is a multiple of 27 plus the sum

$$10 + 11 - 12 + 13 + 14 - 15 + 16 + 17 - 18$$
$$+ 11 + 12 - 13 + 14 + 15 - 16 + 17 + 18$$
$$+ 12 + 13 - 14 + 15 + 16 - 17 + 18$$
$$+ 13 + 14 - 15 + 16 + 17 - 18$$
$$+ 14 + 15 - 16 + 17 + 18 \qquad (***)$$
$$+ 15 + 16 - 17 + 18$$
$$+ 16 + 17 - 18$$
$$+ 17 + 18$$
$$+ 18$$

This is because the occurrences of the integers 19 through 2025 in this sum have exactly the same signs as they did in

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 9, 2025 \rangle$$

Rather than computing the sum (***) explicitly, let's deduce its value by using that the sum (**) equals 123.

The pattern of $+$ and $-$ is identical in both (**) and (***).

As well, the integers in the second triangular sum can be obtained by adding 9 to each of the integers in the first triangular sum.

There are $33 +$ signs and $12 -$ signs (counting a $+$ for the initial terms in each sum), so this means the sum in (***) can be obtained from the the sum in (**) by increasing by 33×9 and decreasing by 12×9, for an overall change of $9(33 - 12) = 9 \times 21 = 27 \times 7$, which is a multiple of 27.

Hence, the sum in (***) is 123 plus a multiple of 27.

Thus, the sum

$$\langle 10, 2025 \rangle + \langle 11, 2025 \rangle + \langle 12, 2025 \rangle + \cdots + \langle 18, 2025 \rangle$$

is a multiple of 27 plus 123 plus another multiple of 27, which is 123 plus some multiple of 27.

The same will be true for the sum of the next nine terms in $G_{2025} - G_{2024}$, and then the next nine, and so on. It is important here that there are exactly 2025 terms in this sum, and 2025 is a multiple of 9.

Since $2025 = 9 \times 225$, we conclude that $G_{2025} - G_{2024}$ is equal to 225×123 plus some multiple of 27. However, $225 \times 123 = 9 \times 25 \times 3 \times 41 = 27 \times 1025$, which is itself a multiple of 27.

We conclude that $G_{2025} - G_{2024}$ is the sum of multiples of 27, so is itself a multiple of 27, and the remainder is 0 when it is divided by 27.

Solution 2

In this solution, we will use the notation and some calculations from the solutions to part (b). Specifically, we will explicitly compute

$$\langle 1, 2025 \rangle + \langle 2, 2025 \rangle + \langle 3, 2025 \rangle + \cdots + \langle 2024, 2025 \rangle + \langle 2025, 2025 \rangle$$

by considering the total contribution of each integer k from 1 through 2025.

From Solution 2 to part (b), if k is a multiple of 3, then the total contribution of k is $\dfrac{k^2}{3}$.

If k is one more than a multiple of 3, then the total contribution of k is $\dfrac{k(k+2)}{3}$. If k is 2 more than a multiple of 3, then the total contribution of k is $\dfrac{k(k+4)}{3}$.

The quantity $G_{2025} - G_{2024}$ is equal to the sum of the contributions of k as k ranges over all integers from 1 through 2025.

Every integer is either a multiple of 3, 1 more than a multiple of 3, or 2 more than a multiple of 3. Therefore, we can compute this sum by grouping the integers according to their "type" and using the formulas given on the contest.

The multiples of 3 from 1 through $2025 = 3 \times 675$ are 3×1, 3×2, and so on to 3×675. Thus, the total contribution of the multiples of 3 is

$$\frac{(3 \times 1)^2}{3} + \frac{(3 \times 2)^2}{3} + \frac{(3 \times 3)^2}{3} + \cdots + \frac{(3 \times 675)^2}{3}$$

which can be simplified to

$$\frac{1}{3}\left(3^2(1)^2 + 3^2(2)^2 + 3^2(3)^2 + \cdots + 3^2(675)^2\right)$$

or

$$\frac{3^2}{3}(1^2 + 2^2 + 3^2 + \cdots + 675^2)$$

Using the formula for the sum of the first n perfect squares with $n = 675$, we get that the total contribution of the multiples of 3 is

$$3 \times \frac{675(675 + 1)(2 \times 675 + 1)}{6} = \frac{675(676)(1351)}{2} = 675(338)(1351)$$

The integers that are 1 more than a multiple of 3 are $1 = 3 \times 0 + 1$, $4 = 3 \times 1 + 1$, $7 = 3 \times 2 + 1$, and so on to $2023 = 3 \times 674 + 1$. Thus, we need to find the sum of all expressions of the form $\dfrac{(3m+1)((3m+1)+2)}{3}$ where m ranges over all integers from 0 through 674.

Observe that

$$\frac{(3m+1)((3m+1)+2)}{3} = \frac{(3m+1)(3m+3)}{3} = (m+1)(3m+1) = 3m^2 + 4m + 1$$

Therefore, by similar calculations to the previous case, the total contribution of integers that are 1 more than a multiple of 3 is

$$\left(3(0)^2 + 4(0) + 1\right) + \left(3(1)^2 + 4(1) + 1\right) + \left(3(2)^2 + 4(2) + 1\right) + \cdots + \left(3(674)^2 + 4(674) + 1\right)$$

which can be rearranged to get

$$3(1^2 + 2^2 + 3^2 + \cdots + 674^2) + 4(1 + 2 + 3 + \cdots + 674) + 675 \times 1$$

and after using the formulas given on the contest, this expression can be simplified to get

$$3 \times \frac{674(675)(1349)}{6} + 4 \times \frac{674(675)}{2} + 675 = 675\left(\frac{674(1349)}{2} + 2(674) + 1\right)$$
$$= 675\big(337(1349) + 2(674) + 1\big)$$
$$= 675\big(337(1353) + 1\big)$$

The integers that are 2 more than a multiple of 3 are $3 \times 0 + 2$, $3 \times 1 + 2$, and so on to $3 \times 674 + 2$, or $3m + 2$ for each integer m from 0 through 674. The contribution of $3m + 2$ is

$$\frac{(3m+2)(3m+2+4)}{3} = \frac{(3m+2)(3m+6)}{3} = (m+2)(3m+2) = 3m^2 + 8m + 4$$

The total contribution of all integers that are 2 more than a multiple of 3 is

$$\left(3(0)^2 + 8(0) + 4\right) + \left(3(1)^2 + 8(1) + 4\right) + \left(3(2)^2 + 8(2) + 4\right) + \cdots + \left(3(674)^2 + 8(674) + 4\right)$$

which is equal to

$$3(1^2 + 2^2 + \cdots + 674^2) + 8(1 + 2 + 3 + \cdots + 674) + 675 \times 4$$

and using the given formulas, the total contribution is

$$\begin{aligned}
3 \times \frac{674(675)(1349)}{6} + 8 \times \frac{674(675)}{2} + 4 \times 675 &= 675\left(\frac{674(1349)}{2} + 4(674) + 4\right) \\
&= 675\left(337(1349) + 4(674) + 4\right) \\
&= 675\left(337(1357) + 4\right)
\end{aligned}$$

Combining the total contributions for the multiples of 3, the integers that are 1 more than a multiple of 3, and the integers that are 2 more than a multiple of 3, we have that $G_{2025} - G_{2024}$ is

$$675(338)(1351) + 675\left(337(1353) + 1\right) + 675\left(337(1357) + 4\right)$$

which is a multiple of $675 = 27 \times 25$, and so $G_{2025} - G_{2024}$ is a multiple of 27.