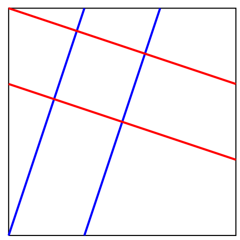
Problem of the Month

Problem 8: May 2023

This month's problem is based on Problem 6 part (b) of the 2023 Euclid contest. Here is a modified and rephrased version of that problem:

A square is drawn in the plane with vertices at $(0, 0), (1, 0), (1, 1)$, and $(0, 1)$. Two blue lines are drawn with slope 3, one passing through $(0, 0)$ and the other through $(\frac{1}{3}, 0)$. Two red lines are drawn with slope $-\frac{1}{3}$, one passing through $(0, 1)$ and the other through $(0, \frac{2}{3})$. What is the area of the square bounded by the red and blue lines?



The answer to this question is $\frac{1}{10}$. We suggest convincing yourself of this before attempting the rest of the problem. The fact that this small square has an area exactly one tenth of the larger square suggests that there is a way to answer this question by showing that exactly 10 squares the size of the small one should fit into the large square. Let's explore!

Here is some terminology that we will use in this problem.

Definition: A *lattice point* is a point (x, y) for which x and y are both integers.

Definition: A 1×1 square whose vertices are at lattice points is called a *unit lattice square*. We will denote by T the unit lattice square with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$, and $(0, 1)$.

Definition: $L_{q,p}$ denotes the line segment connecting $(0, 0)$ to the point (q, p). Note that this line has slope $\frac{p}{q}$.

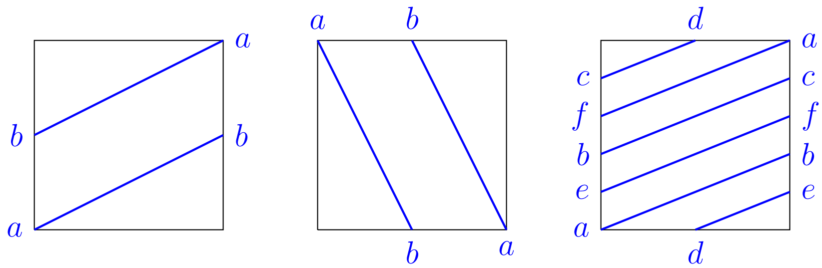
Definition: An $m-$lattice line is a line with slope m that passes through at least one lattice point.

The next two definitions are more complicated. There are examples given after they are stated.

Definition: The *tricky unit square*, $\mathbb{T}$, is a modified version of T (see above) with the property that if a line reaches an edge of $\mathbb{T}$, it "jumps" to the opposite side and continues with the same slope. For example, if a line reaches the top edge of $\mathbb{T}$, it continues with the same slope from the bottom edge directly below where it reached the top edge. If a line reaches a vertex of $\mathbb{T}$, then it has simultaneously reached two edges. In this situation, it continues with the same slope from the opposite vertex of $\mathbb{T}$.

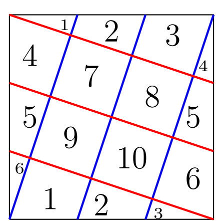
Definition: Let $\frac{p}{q}$ be a rational number written in lowest terms. The $\frac{p}{q}-$*loop* on $\mathbb{T}$ is the line passing through $(0, 0)$ with slope $\frac{p}{q}$.

Below are diagrams, from left to right, of the $\frac{1}{2}-$loop, the $\frac{-2}{1}-$loop, and the $\frac{2}{5}-$loop on $\mathbb{T}$. In each diagram, equal letters mark places where the line jumps from one side of the square to the opposite side.



Each of the loops above eventually come back to their starting point and repeat. This happens because $\frac{p}{q}$ is rational (think about this!). Notice that in the image of the $\frac{-2}{1}-$loop (the middle image), the loop starts at $(0,1)$ instead of $(0,0)$. This is because a line of negative slope starting at $(0,0)$ (on the bottom edge) immediately jumps to the top edge and continues from $(0,1)$. It is worth thinking about how all four vertices of $\mathbb{T}$ really represent the same point.

Below is an image of $\mathbb{T}$ with the $\frac{3}{1}-$loop in blue and the $\frac{-1}{3}-$loop in red. These two loops divide $\mathbb{T}$ into 10 smaller squares. The squares numbered 1, 2, 3, 4, 5, and 6 are split in two pieces each across edges of $\mathbb{T}$. Notice that both the loops pass thorough the point $(0,0)$ since the four vertices of the square are the same point. This means, if we count the four vertices as one intersection point, the two loops intersect exactly 10 times (count them!). Think about how this compares to the Euclid problem mentioned earlier.



(a) For each pair of loops below, draw $\mathbb{T}$ with that pair of loops, count the number of times the loops intersect, and count the number of squares into which the loops divide $\mathbb{T}$.

  (i) The $\frac{4}{1}-$loop and the $\frac{-1}{4}-$loop.

  (ii) The $\frac{2}{3}-$loop and the $\frac{-3}{2}-$loop.

  (iii) The $\frac{4}{3}-$loop and the $\frac{-3}{4}-$loop.

  (iv) The $\frac{1}{1}-$loop and the $\frac{-1}{1}-$loop.

In the remaining problems, p and q are positive integers that have no positive divisors in common other than 1.

(b) How many line segments make up the $\frac{p}{q}-$loop?

(c) How many $\frac{p}{q}-$lattice lines intersect T?

(d) Explain why the number of points of intersection of the $\frac{p}{q}$–loop and the $-\frac{q}{p}$–loop on $\mathbb{T}$ is equal to the number of small squares into which the loops divide $\mathbb{T}$. Remember that the four vertices of $\mathbb{T}$ represent the same point.

(e) How many $-\frac{q}{p}$–lattice lines intersect $L_{q,p}$?

(f) Compute the area of the small squares in $\mathbb{T}$ created by the $\frac{p}{q}$–loop and the $-\frac{q}{p}$–loop. Our solution will take for granted that these small squares all have the same area, but you might like to think about how to prove this.