



Problem of the Month

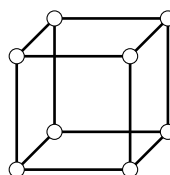
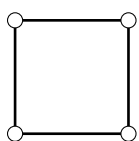
Problem 7: April 2023

Let A_n denote the set of all n -tuples of 0s and 1s. For example, A_2 is the set of all ordered pairs of 0s and 1s or $A_2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. To improve readability, we will omit the commas and parentheses from the elements of A_n . For example, the elements of A_3 will be denoted by 000, 001, 010, 011, 100, 101, 110, and 111.

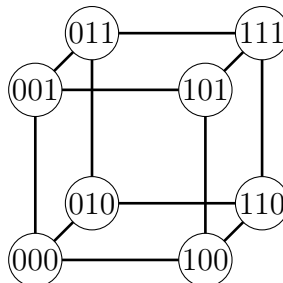
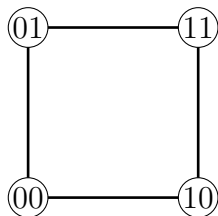
Variables referring to elements of A_n will be bold lowercase letters. For example, we might refer to elements in A_2 as \mathbf{a} , or \mathbf{b} , and so on. To refer to the coordinates of elements in A_n , we will use square brackets. For example, if $\mathbf{a} = 11010$, an element in A_5 , then $\mathbf{a}[1] = 1$, $\mathbf{a}[2] = 1$, $\mathbf{a}[3] = 0$, $\mathbf{a}[4] = 1$, and $\mathbf{a}[5] = 0$.

For two elements \mathbf{a} and \mathbf{b} in A_n , the *Hamming distance*, denoted $d(\mathbf{a}, \mathbf{b})$ between \mathbf{a} and \mathbf{b} is equal to the number of coordinates where they differ. For example, if $\mathbf{a} = 11010$ and $\mathbf{b} = 01110$, then $d(\mathbf{a}, \mathbf{b}) = 2$ because $\mathbf{a}[1] \neq \mathbf{b}[1]$ and $\mathbf{a}[3] \neq \mathbf{b}[3]$, but $\mathbf{a}[i] = \mathbf{b}[i]$ for $i = 2$, $i = 4$, and $i = 5$. It is important to convince yourself that $d(\mathbf{a}, \mathbf{b}) = d(\mathbf{b}, \mathbf{a})$ for any \mathbf{a} and \mathbf{b} .

The notion of a *graph* was defined in the extra information about the February 2023 problem, but there are also plenty of places online that have definitions. We will keep things simple here and define a graph to be a collection of *vertices*, some of which are connected to each other by *edges*. When we draw a graph, a vertex will be represented by a circle and an edge will be represented by a line segment from one vertex to another. Two examples of graphs are depicted below. The one on the left has four vertices and the one on the right has eight. Note that two edges intersecting does not necessarily imply the presence of a vertex.



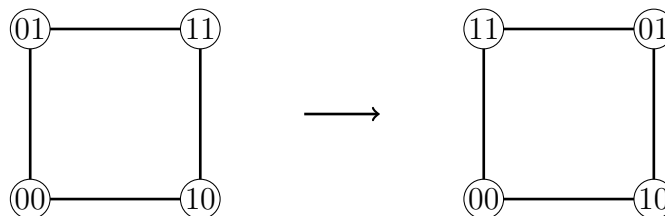
For each n , we define a graph with 2^n vertices called the *natural graph* of A_n . In the natural graph of A^n , it is possible to label every vertex by exactly one element of A_n such that there is an edge between two vertices exactly when their Hamming distance is 1. The two examples above are the natural graphs of A_2 and A_3 . They are shown again below with their vertices labelled.





A *walk* in a graph from vertex v to vertex w is a sequence of vertices starting at v and ending at w with the property that there is an edge connecting every pair of consecutive vertices in the sequence. The *length* of a walk is the number of edges it uses. For example, let v , w , x , and y be the vertices labelled by 000, 110, 100, and 010 in the natural graph of A_3 , respectively. Then v, x, w and v, y, w are walks of length 2 from v to w . The *distance* between v and w in a graph is equal to the shortest possible length of a walk from v to w . In the example above, v and w have a distance of 2 because there are walks of length 2, but there are no shorter walks from v to w .

- Let \mathbf{a} and \mathbf{b} be elements of A_n . Show that $d(\mathbf{a}, \mathbf{b})$ is equal to the distance between \mathbf{a} and \mathbf{b} in the natural graph of A_n .
- For each k with $1 \leq k \leq n$, determine the number of two-element subsets $\{\mathbf{a}, \mathbf{b}\}$ of A_n that satisfy $d(\mathbf{a}, \mathbf{b}) = k$.
- Suppose we were to relabel the vertices of the natural graph of A_n by permuting the labels. That is, we keep the graph the same but use the elements of A_n to label a vertex of the graph in some other way. For example, in A_2 , we might leave 00 and 10 where they are and swap the positions of 01 and 11, as shown.



When this is done, the distance in the new graph between elements of \mathbf{a} and \mathbf{b} is not necessarily equal to $d(\mathbf{a}, \mathbf{b})$ any more. The table below has the two-element subsets of A_2 in the first column, $d(\mathbf{a}, \mathbf{b})$ in the second column, and their distance in the relabelled graph in the third column.

$\{\mathbf{a}, \mathbf{b}\}$	$d(\mathbf{a}, \mathbf{b})$	new distance
$\{00, 01\}$	1	2
$\{00, 10\}$	1	1
$\{00, 11\}$	2	1
$\{01, 10\}$	2	1
$\{01, 11\}$	1	1
$\{10, 11\}$	1	2

Among the four subsets $\{\mathbf{a}, \mathbf{b}\}$ with $d(\mathbf{a}, \mathbf{b}) = 1$, there are two that have a distance in the relabelled graph of 1 and two that have a distance in the relabelled graph of 2.

Now for the question: For each n , find a way to permute the elements of A_n so that the following happens: Among all two-element subsets $\{\mathbf{a}, \mathbf{b}\}$ of A_n with $d(\mathbf{a}, \mathbf{b}) = 1$, there are the same number with each possible distance in the relabelled graph.