Problem of the Month
Problem 5: February 2023

Hint

(a) Remember the condition that no prime number can divide every integer in a splendid sequence.

(b) Think of an integer that is a divisor of every integer.

(c) Try the positive integer $c = a_i + a_{i+1}$.

(d) Prove that if the divisibility conditions (S1 and S2 in the problem statement) are satisfied by $(a_1, a_2, a_3, \ldots, a_n)$, then a_1 and a_n must both divide every integer in the sequence.

(e) Try to write down a few splendid sequences of length at least 5. With the hint for (c) in mind, what do you notice about the largest integer in a splendid sequence? Show that, for each n, there is a largest possible value that an integer in any splendid sequence of length n can take. For instance, it can be shown that in a splendid sequence of length $n \geq 2$, every integer must be less than 2^{n-2}.

(f) There is a famous sequence called the *Catalan numbers* where the n^{th} Catalan number equal to $\dfrac{1}{n+1}\dbinom{2n}{n}$. The Catalan numbers arise in many interesting ways in mathematics. One such way is that the n^{th} Catalan number is equal to the number of sequences $(b_1, b_2, \ldots, b_n)$ of length n with the following properties.

- Each b_k is a positive integer.
- $b_1 = 1$.
- For each k satisfying $1 \leq k \leq n-1$, $b_{k+1} \leq b_k + 1$. That is, b_{k+1} is no more than one more than b_k (it is allowed to be less than or equal to b_k).

In the solution, such sequences will be called *tame sequences*. One way to answer this question is to show that the number of tame sequences of length $n-1$ is equal to the number of splendid sequences of length n and use the closed form for the number of tame sequences. To do this, we suggest trying to devise a way to use a tame sequence of length $n-1$ as "instructions" to construct a splendid sequence of length n. The idea from part (c) will probably be important.

Note: In the solution, we will provide a proof that the number of tame sequences of length n is the n^{th} Catalan number. You might want to try to prove this yourself, but we recommend taking it for granted when trying to solve part (f).