# Problem of the Month
## Problem 4: January 2023

## Hint

(a) What are the possible values of $c$?

(b) How many distinct integers can occur in a triple in $S$?

(c) Try to generalize the idea in part (c). The constants $a_2$ through $a_{k+1}$ do not depend on $n$.

(d) For positive integers $u$ and $v$ with $u < v$, the usual convention is that $\binom{u}{v} = 0$. This convention makes sense for (at least) two reasons. First, there are zero ways to choose $v$ objects from $u$ distinct objects if $u < v$, so "$u$ choose $v$" should be equal to 0. Second, the formula for $\binom{u}{v}$ given by

$$\binom{u}{v} = \frac{u(u-1)(u-2)\cdots(u-v+1)}{v!}$$

will have a factor of 0 in the numerator if $u < v$.

(e) Directly compute an expression for $p_5(n) - p_5(n-1)$. It should be a polynomial with coefficients depending on $a_1$ through $a_6$. By equating coefficients with the polynomial $n^5$, solve for $a_1$ through $a_6$. After these coefficients are known, $a_0$ can be computed from $p_5(1) = 1$.

(f) A polynomial with infinitely many roots must be the constant zero polynomial. Using this fact, show that $p_k(n) - p_k(n-1) = n^k$ for all real numbers, not just positive integers. This means you need to "extend" $p_k(n)$ to accept inputs that are not positive integers. Once this is done, determine the values of $p_k(0)$ and $p_k(-1)$. To show that $2n+1$ is a factor of $p_k(n)$ for even $k$, consider the values of $p_k(-n)$ when $n$ is a positive integer.