Problem of the Month

Problem 3: December 2023

This month's problem is an extension of Problem 3 from Part B of the 2023 Canadian Intermediate Mathematics Contest. The original problem was stated as follows:

> The positive integers are written into rows so that Row n includes every integer m with the following properties:
>
> (i) m is a multiple of n,
>
> (ii) $m \leq n^2$, and
>
> (iii) m is not in an earlier row.
>
> The table below shows the first six rows.
>
> | | |
> |---|---|
> | Row 1 | 1 |
> | Row 2 | 2, 4 |
> | Row 3 | 3, 6, 9 |
> | Row 4 | 8, 12, 16 |
> | Row 5 | 5, 10, 15, 20, 25 |
> | Row 6 | 18, 24, 30, 36 |
>
> (a) Determine the smallest integer in Row 10.
>
> (b) Show that, for all positive integers $n \geq 3$, Row n includes each of $n^2 - n$ and $n^2 - 2n$.
>
> (c) Determine the largest positive integer n with the property that Row n does not include $n^2 - 10n$.

If you have not already done so, we suggest thinking about the parts above before proceeding.

(a) For each positive integer k, determine the largest positive integer n with the property that Row n does not include $n^2 - kn$. (This generalizes part (c) from the original problem.)

In the remaining questions, $f(n)$ is defined for each $n \geq 1$ to be the largest non-negative integer m such that $m \leq n$ and mn is *not* in Row n. For example, Row 6 is $18, 24, 30, 36$, so $f(6) = 2$ since $2 \times 6 = 12$ is not in Row 6 but 3×6, 4×6, 5×6, and 6×6 are all in Row 6.

(b) Show that $f(p) = 0$ for all prime numbers p. (Looking closely at the definition of $f(n)$, $f(p) = 0$ means that every positive multiple of p from p through p^2 appears in Row p.)

(c) Find an expression for $f(pq)$ where p and q are prime numbers. Justify that the expression is correct.

(d) Find an expression for $f(p^d)$ where p is a prime number and d is a positive integer.

(e) Take some time to explore the function f further on your own. Are there other results you can prove about the function beyond what is done in (b), (c) and (d)? Is there a nice way to compute $f(n)$ in general without computing each of the first $n - 1$ rows?