



## Problem of the Month

### Solution to Problem 1: October 2023

Before presenting solutions to the various parts of this question, we present a few general facts that will be used throughout the solutions. As well, we will often denote a list of integers enclosed in brackets. For example, we might write  $[1, 2, 3, 4, 5]$  instead of  $1, 2, 3, 4, 5$ . This is to emphasize when we want to think of a list as a single object to which we can refer.

**Fact 1:** If  $n$  is a positive integer with units digit different from 9, then  $D(n + 1) = D(n) + 1$ .

*Proof.* Suppose that  $n$  is a  $k$ -digit positive integer with digits  $A_1, A_2, \dots, A_{k-1}, A_k$ , when read from left to right. That is,  $n$  is  $A_1A_2 \cdots A_{k-1}A_k$ . Then  $D(n) = A_1 + A_2 + \cdots + A_{k-1} + A_k$ . Since  $A_k \neq 9$ , no “carry” occurs when adding 1 to  $n$ . In other words,  $n + 1$  is a  $k$ -digit positive integer with digits  $A_1, A_2, \dots, A_{k-1}, A_k + 1$  when read from left to right. Therefore,  $D(n + 1) = A_1 + A_2 + \cdots + A_{k-1} + (A_k + 1) = (A_1 + A_2 + \cdots + A_{k-1} + A_k) + 1 = D(n) + 1$ .  $\square$

**Fact 2:** Suppose that  $L = [n, n + 1, n + 2, \dots, n + k]$  is a list of consecutive non-negative integers. If none of  $n + 1, n + 2, \dots, n + k$  is a multiple of 10, then the integers  $D(n), D(n + 1), \dots, D(n + k)$  are consecutive.

*Proof.* None of  $n, n + 1, \dots, n + k - 1$  has a units digit of 9. This is because the alternative would imply that one of  $n + 1, n + 2, \dots, n + k$  has a units digit of 0 and so is a multiple of 10. Therefore, we can repeatedly apply Fact 1 to get that  $D(n + 1)$  is one more than  $D(n)$ ,  $D(n + 2)$  is one more than  $D(n + 1)$ , and so on up to  $D(n + k)$  is one more than  $D(n + k - 1)$ .  $\square$

**Fact 3:** Suppose  $m$  is an integer with  $m > 1$  and that  $[a, a + 1, a + 2, \dots, a + (m - 2)]$  is a list of  $m - 1$  consecutive integers with the property that no integer in the list is a multiple of  $m$ . Then the remainder after  $a$  is divided by  $m$  is 1 and the remainder after  $a + m - 2$  is divided by  $m$  is  $m - 1$ .

*Proof.* In the hint, it was stated that the remainders after dividing the positive integers  $1, 2, 3, 4, 5, \dots$  by  $m$  are

$$1, 2, 3, \dots, m - 2, m - 1, 0, 1, 2, 3, \dots, m - 2, m - 1, 0, 1, 2, \dots$$

That is, the remainders cycle through the values  $1, 2, 3, \dots, m - 2, m - 1, 0$  in that order.

Consider the list  $[a, a + 1, a + 2, \dots, a + (m - 2)]$ . Since none of the  $m - 1$  integers in this list are multiples of  $m$ , none of the remainders after dividing by  $m$  can be 0, so the remainders must be  $1, 2, 3, \dots, m - 2, m - 1$  in that order. In particular, the remainder after dividing  $a$  by  $m$  is 1 and the remainder after dividing  $a + (m - 2)$  by  $m$  is  $m - 1$ .  $\square$

For a positive integer  $n$  with units digit  $d$ , we say that  $n$  has  $k$  trailing  $d$ s if the rightmost  $k$  digits of  $n$  are  $d$  but the rightmost  $k + 1$  digits are not equal to  $d$ . For example, the integer 123 has one trailing 3, 45000 has three trailing 0s, and 29199 has two trailing 9s.

By Fact 1, it is “usually” easy to predict  $D(n + 1)$  from  $D(n)$  since if the units digit of  $n$  is not 9, then  $D(n + 1) = D(n) + 1$ . However, if the units digit of  $n$  is 9, then adding 1 causes a carry to happen, so the digit sum can go down. For example, the digit sum of 2499 is  $D(2499) = 2 + 4 + 9 + 9 = 24$ , but the digit sum of  $2499 + 1 = 2500$  is  $D(2500) = 2 + 5 + 0 + 0 = 7$ . Fact 4 establishes a relationship between  $D(n)$  and  $D(n + 1)$  when  $n$  has a units digit of 9.

**Fact 4:** Suppose  $n$  is a positive integer with  $t$  trailing 9s. Then  $D(n + 1) - D(n) = 1 - 9t$ .

*Proof.* Suppose  $n$  has  $t$  trailing 9s, which means we can assume that  $n$  takes the form  $n = A_1A_2A_3 \cdots A_{r-1}A_r99 \cdots 99$  where the  $A_i$  are the first  $r$  digits and  $A_r \neq 9$ . Since  $A_r$  is a digit that is not 9, it must be less than 9. Therefore, when we add 1 to  $n$ , the first  $r - 1$  digits do not change, the  $r^{\text{th}}$  digit increases by 1, and all of the trailing 9s become trailing 0s. In symbols,  $n + 1$  is the integer  $A_1A_2 \cdots A_{r-1}(A_r + 1)00 \cdots 00$ .

Computing digit sums,  $D(n) = A_1 + A_2 + \cdots + A_r + 9t$  and  $D(n + 1) = A_1 + A_2 + \cdots + A_{r-1} + (A_r + 1)$ . Subtracting, we get cancellation of  $A_1$  through  $A_r$  and are left with  $D(n + 1) - D(n) = 1 - 9t$ .  $\square$

**Remark:** We will repeatedly use the observation that if  $L = [a, a + 1, a + 2, \dots, a + k]$  is an  $m$ -list for some  $m$ , then every sublist of  $L$  is also an  $m$ -list. One important consequence of this fact is that if we can show that there does not exist an  $m$ -list of length  $k$ , then we can conclude that there is no  $m$ -list that is longer than  $k$ . Thus, in general, if we want to prove that  $k$  is the maximum length of an  $m$ -list, it suffices to do the following:

- Produce an example of an  $m$ -list of length  $k$ .
  - Show that there does not exist an  $m$ -list of length  $k + 1$ .
- (a) The list  $[9, 10]$  has  $D(9) = 9$  and  $D(10) = 1$ , both of which are odd. Therefore,  $[9, 10]$  is a 2-list of length 2. We will now show that a list of three consecutive positive integers cannot be a 2-list.

Suppose  $a$  is a positive integer with the property that  $D(a)$  and  $D(a + 1)$  are both odd. Since  $D(a)$  is odd,  $D(a) + 1$  is even. We are assuming that  $D(a + 1)$  is odd, and since  $D(a) + 1$  is even, we conclude that  $D(a + 1) \neq D(a) + 1$ .

By Fact 1, the units digit of  $a$  must be 9 (otherwise  $D(a + 1) = D(a) + 1$ ), so the units digit of  $a + 1$  is 0. Again by Fact 1,  $D(a + 2) = D(a + 1) + 1$ , and since  $D(a + 1)$  is odd,  $D(a + 1) + 1 = D(a + 2)$  is even.

Now consider the list  $[a, a + 1, a + 2]$ . If  $D(a)$  and  $D(a + 1)$  are both odd, we have shown that  $D(a + 2)$  is even. This means it is impossible for  $D(a)$ ,  $D(a + 1)$ , and  $D(a + 2)$  to all be odd, so we conclude that a 2-list of length 3 cannot exist.

- (b) The arguments presented in this part and in part (d) rely on examining the location of multiples of 10 in lists of consecutive integers. For this part, because of Fact 2, seven consecutive integers will have seven consecutive digit sums unless a multiple of 10 causes a carry. To build a 7-list of length more than six, there needs to be a multiple of 10 somewhere in the middle to avoid ever having seven consecutive digit sums. You might be able to guess from this that the maximum length should be around 12.

Suppose  $L = [a, a + 1, a + 2, \dots, a + 12]$  is an arbitrary list of 13 consecutive non-negative integers. We will show that  $L$  cannot be a 7-list and this will show that a 7-list cannot have length 13.

Since  $L$  contains 13 consecutive integers, it must contain at least one multiple of 10. Thus, there must be some  $k$  with  $0 \leq k \leq 12$  such that  $a + k$  is a multiple of 10.

**Case 1:**  $0 \leq k \leq 6$ .

Since  $k \leq 6$ ,  $a + k + 6 \leq a + 12$ . The list  $L$  contains the integers  $a$  through  $a + 12$ , so we conclude that the seven integers  $a + k, a + k + 1, \dots, a + k + 6$  are all in  $L$ . Moreover, the smallest multiple of 10 after  $a + k$  is  $a + k + 10$ , so none of  $a + k + 1, a + k + 2, \dots, a + k + 6$  is a multiple of 10 since they are all strictly between two consecutive multiples of 10.

By Fact 2, the integers  $D(a + k), D(a + k + 1), \dots, D(a + k + 6)$  are consecutive. Any list of seven consecutive integers must contain a multiple of 7, so we conclude that  $L$  is not a 7-list.

**Case 2:**  $7 \leq k \leq 12$ .

Since  $7 \leq k$ ,  $a + 7 \leq a + k$  which can be rearranged to get  $a \leq a + k - 7$ . By similar reasoning to the beginning of Case 1, the seven integers  $a + k - 7, a + k - 6, \dots, a + k - 1$  are all in  $L$ . As well, they are strictly between  $a + k - 10$  and  $a + k$ , which are consecutive multiples of 10. Therefore, none of  $a + k - 7, a + k - 6, \dots, a + k - 1$  is a multiple of 10.

By Fact 2,  $D(a + k - 7), D(a + k - 6), \dots, D(a + k - 1)$  are consecutive, so one of them must be a multiple of 7. Therefore,  $L$  is not a 7-list.

Cases 1 and 2 address all possibilities for the location of a multiple of 10 in  $L$ . In both cases,  $L$  is not a 7-list. We conclude that a list of 13 consecutive non-negative integers cannot be a 7-list.

To help find a 7-list of length 12, we can modify the reasoning above slightly. To that end, suppose  $M = [a, a + 1, a + 2, \dots, a + 11]$  is a 7-list. In the previous paragraphs, we essentially used the fact that if seven consecutive non-negative integers do *not* include a multiple of 10, then one of their digit sums must be a multiple of 7. Therefore, for  $M$  to be a 7-list, every sublist of seven-consecutive integers must contain a multiple of 10. We leave it to the reader to verify that this forces  $a + 6$  to be a multiple of 10.

Since  $a + 6$  is a multiple of 10, the closest multiples of 10 to  $a + 6$  are  $a - 4$  and  $a + 16$ . Therefore, the list  $a, a + 1, \dots, a + 5$  does not contain a multiple of 10, and the only multiple of 10 in the list  $a + 6, a + 7, \dots, a + 11$  is  $a + 6$ . By Fact 2,

$$D(a), D(a + 1), D(a + 2), D(a + 3), D(a + 4), D(a + 5)$$

and

$$D(a + 6), D(a + 7), D(a + 8), D(a + 9), D(a + 10), D(a + 11)$$

are lists of six consecutive integers. We are assuming  $M$  is a 7-list, so no integer in either of these lists is a multiple of 7. By Fact 3, the remainder after dividing  $D(a + 5)$  by 7 is  $7 - 1 = 6$ . Also by Fact 3, the remainder after dividing  $D(a + 6)$  by 7 is 1.

To summarize, if  $M = [a, a + 1, \dots, a + 11]$  is a 7-list, then the following must be true.

- $a + 6$  is a multiple of 10.
- The remainder after dividing  $D(a + 5)$  by 7 is 6.
- The remainder after dividing  $D(a + 6)$  by 7 is 1.

The first condition implies that  $a + 5$  has a units digit of 9, so suppose that  $n + 5$  has  $t$  trailing 9s. By Fact 4,  $D(n + 6) - D(n + 5) = 1 - 9t$ . The second two conditions imply that there are integers  $m$  and  $n$  such that  $D(a + 5) = 7n + 6$  and  $D(a + 6) = 7m + 1$ . Substituting into  $D(a + 6) - D(a + 5) = 1 - 9t$ , we obtain  $(7n + 1) - (7m + 6) = 1 - 9t$  which can be rearranged to get  $2t - 6 = 7(m - n - t)$ . We know that  $t$  is a positive integer, and this shows that  $2t - 6$  must be a multiple of 7. The smallest positive integer  $t$  with the property that  $2t - 6$  is a multiple of 7 is  $t = 3$  since  $2(3) - 6 = 0$ , which is a multiple of 7.

We are looking for an integer  $a$  such that  $a + 5$  ends in three 9s and has the property that  $D(n + 5)$  is six more than a multiple of 7. Thus,  $a + 5$  will have the form  $x999$  where  $x$  is some string of digits. In fact,  $D(999) = 9 + 9 + 9 = 27$  has a remainder of 6 when divided by 7, so we set  $a + 5 = 999$  or  $a = 994$ . Indeed, if we set

$$M = [994, 995, 996, 997, 998, 999, 1000, 1001, 1002, 1003, 1004, 1005]$$

the digit sums of the integers in  $M$  are 22, 23, 24, 25, 26, 27, 1, 2, 3, 4, 5, 6, none of which is a multiple of 7. Therefore,  $M$  is a 7-list of length 12.

This is actually the earliest 7-list of length 12. Others can be found by prepending digits to 994 in such a way that the number of trailing 9s of  $a + 5$  does not change, and the remainder after dividing the digit sum by 7 does not change. This can be done by prepending any digits that have a sum that is a multiple of 7, provided the rightmost new digit is not a 9.

For example, we could take  $a$  to be 7994, 16994, 25994, 662994, and so on.

- (c) Following the reasoning from (b), one might think that we can find a 9-list of length  $8 + 8 = 16$ . Specifically, we might expect that we can find a list of the form  $[a, a + 1, \dots, a + 15]$  where  $a + 8$  is a multiple of 10 so that there are never nine consecutive integers without a multiple of 10. In fact, the maximum length of a 9-list is eight, so the approach in part (b) cannot be applied here. You might like to try to follow the reasoning from part (b) through to see exactly what goes wrong. The proof given here uses the well-known fact that a positive integer  $n$  is a multiple of 9 if and only if  $D(n)$  is a multiple of 9.

A list of nine consecutive integers must contain a multiple of 9, so it must contain an integer whose digit sum is a multiple of 9. Therefore, a list of 9 consecutive integers cannot be a 9-list.

Any list of eight consecutive integers, none of which is a multiple of 9, must be a 9-list since none of the digit sums are multiples of 9. An example of a 9-list of length 8 is  $[1, 2, 3, 4, 5, 6, 7, 8]$ .

- (d) It might come as a surprise, but the maximum length of an 11-list is 38.

To get an idea of how one might guess this, we will first determine the maximum length of an 11-list that starts with a multiple of 10.

Suppose  $L = [a, a + 1, a + 2, \dots, a + 19]$  is an 11-list of length 20 with the property that  $a$  is a multiple of 10. By Fact 2,  $D(a), D(a + 1), \dots, D(a + 9)$  and  $D(a + 10), D(a + 11), \dots, D(a + 19)$  are lists of ten consecutive integers. Since we are assuming that  $L$  is an 11-list, no integer in either of these lists is a multiple of 11. Therefore, by Fact 3, the remainder after  $D(a + 9)$  is divided by 11 is 10 and the remainder after  $D(a + 10)$  is divided

by 11 is 1. This implies the existence of integers  $m$  and  $n$  such that  $D(a + 9) = 11m + 10$  and  $D(a + 10) = 11n + 1$ .

Since  $a$  is a multiple of 10, the units digit of  $a + 9$  is 9. Suppose  $a + 9$  has  $t$  trailing 9s. By Fact 4,  $D(a + 10) - D(a + 9) = 1 - 9t$ . Substituting  $D(a + 9) = 11m + 10$  and  $D(a + 10) = 11n + 1$  into this equation, we get

$$(11n + 1) - (11m + 10) = 1 - 9t$$

which can be rearranged to get  $9t = 11(m - n) + 10$ . This means  $t$  is a positive integer with the property that  $9t$  is ten more than a multiple of 11. It can be checked that  $t = 6$  is the smallest positive integer with this property.

We have shown the following: if  $[a, a + 1, a + 2, \dots, a + 19]$  is an 11-list of length 20 with the property that  $a$  is a multiple of 10, then  $a + 9$  has at least six trailing 9s. We can use this fact to prove the following:

- (i) An 11-list that starts with a multiple of 10 cannot have length 30.
- (ii) An 11-list cannot have length 39.

To prove (i), assume that  $L = [a, a + 1, a + 2, \dots, a + 29]$  is an 11-list of length 30 and that  $a$  is a multiple of 10. We will show that this implies something that is plainly false, which will allow us to conclude that such an 11-list cannot exist.

The fact that  $L$  is an 11-list implies that the lists  $M = [a, a + 1, \dots, a + 19]$  and  $N = [a + 10, a + 11, \dots, a + 29]$  are both 11-lists. Moreover,  $a$  is a multiple of 10, so  $a + 10$  is a multiple of 10. Thus,  $M$  and  $N$  are both 11-lists of length 20 that start with a multiple of 10. By what was established earlier, both  $a + 9$  and  $a + 10 + 9 = a + 19$  have at least six trailing 9s.

The difference between two integers with at least six trailing 9s must have at least six trailing 0s. However,  $(a + 19) - (a + 9) = 10$  has only one trailing 0. We have shown that if an 11-list starts with a multiple of 10 and has length 30, then 10 has at least six trailing 0s. Since 10 has only one trailing 0, we conclude that it must be impossible for an 11-list to have length 30 and start with a multiple of 10. **Note:** The technique just used is known as a *proof by contradiction*.

To prove (ii), we can apply (i). Suppose  $L = [a, a + 1, a + 2, \dots, a + 38]$  is a list of 39 consecutive positive integers. Among the first ten integers in  $L$ , there must be a multiple of 10. Suppose  $k$  is an integer with  $0 \leq k \leq 9$  such that  $a + k$  is a multiple of 10. Observe that  $a + k + 29 \leq a + 9 + 29 = a + 38$ , which means the list  $M = [a + k, a + k + 1, \dots, a + k + 29]$  is completely contained in  $L$ . The list  $M$  has length 30 and starts with a multiple of 10, so it cannot be an 11-list by (i). Since  $L$  contains a sub-list that is not an 11-list, it cannot be an 11-list. Therefore, an 11-list cannot have length 39.

We can now use what we have done to find an 11-list of length 38. Suppose that  $L = [a, a + 1, a + 2, \dots, a + 37]$  is an 11-list. There must be an integer  $k$  with  $0 \leq k \leq 9$  such that  $a + k$  is a multiple of 10. If  $0 \leq k \leq 8$ , then  $a + k + 29 \leq a + 37$ , so  $[a + k, a + k + 1, \dots, a + k + 29]$  is entirely contained in  $L$ . This would imply the existence of an 11-list of length 30 that starts with a multiple of 10, which is impossible by (i). Since  $0 \leq k \leq 8$  is impossible, we conclude that  $k = 9$ .

We assumed that  $L = [a, a + 1, a + 2, \dots, a + 37]$  was an 11-list and deduced that  $a + 9$  is a multiple of 10. The list  $[a + 9, a + 10, \dots, a + 28]$  is completely contained in  $L$ , has length 20, and starts with a multiple of 10. Therefore, from the argument at the beginning of the solution to this part,  $a + 9 + 9 = a + 18$  has at least six trailing 9s. As well, Since  $a + 9$  is a multiple of 10, we can apply Fact 2 to  $a + 9, a + 10, \dots, a + 18$  to get that  $D(a + 9), D(a + 10), \dots, D(a + 18)$  is a list of 10 consecutive integers. Since  $L$  is an 11-list, none of them is a multiple of 11. By Fact 3, the remainder after  $D(a + 18)$  is divided by 11 is 10.

Therefore, we will look for an integer  $a$  with the property that  $a + 18$  has 6 trailing 9s and  $D(a + 18)$  is 10 more than a multiple of 11. The integer 999999 has  $D(999999) = 54$  which is 10 more than a multiple of 11, so we guess that  $a + 18 = 999999$ , or  $a = 999981$ . It can be checked that the list  $[999981, 999982, \dots, 1000018]$  is an 11-list of length 38. It also follows from our reasoning that this is the first such 11-list.