



Problem of the Month

Problem 1: October 2023

Hint

For an integer a , think about how $D(a)$ and $D(a + 1)$ compare to each other. The relationship is most interesting when $a + 1$ is a multiple of 10.

Our solution will make use of some basic properties of remainders. For positive integers a and b , the remainder when dividing a by b can be found by first determining qb , the greatest multiple of b with $qb \leq a$, then computing the remainder as $r = a - qb$. The integer q can be found by computing $\frac{a}{b}$ and rounding down. For example, the remainder when dividing 38 by 5 is 3 because 35 is the greatest multiple of 5 that does not exceed 38 (in this case, $q = 7$), and so the remainder is $38 - 35 = 3$. We assume this idea is familiar, but here are a couple of things to think about and keep in mind.

- For integers a and b with $b > 0$, the remainder, r , when dividing a by b satisfies $0 \leq r < b$.
- For a fixed positive integer b , the remainders when dividing the positive integers $1, 2, 3, 4, 5, \dots$ cycle through the integers $1, 2, 3, \dots, b - 2, b - 1, 0$ repeatedly in that order. Importantly, if we think of 0 as coming “after” $b - 1$, the remainders for consecutive integers are consecutive.
- The remainder when dividing a by b is 0 exactly when a is a multiple of b .

If you have seen modular arithmetic before, it could be useful in writing a solution. Our solution will not use modular arithmetic, but we suggest reading about it anyway since it is quite useful.

Below are some specific hints for the given questions.

- (a) Show that if $D(a)$ and $D(a + 1)$ are both odd, then $a + 1$ is a multiple of 10.
 - (b) First, try to find the maximum length of a 7-list that does not contain a multiple of 10. For an integer a , how do the remainders when $D(a)$ and $D(a + 1)$ are divided by 7 relate to each other?
 - (c) By a rather famous divisibility rule, a positive integer is a multiple of 9 if and only if the sum of its digits is a multiple of 9.
 - (d) First show that an 11-list that starts with a multiple of 10 has length at most 29. Think about the remainders when $D(a)$ and $D(a + 1)$ are divided by 11.
-
-