Problem of the Month

Solution to Problem 0: September 2023

(a) (i)

$$\begin{aligned} r_2 = f(r_1) &= \frac{2r_1 - 1}{r_1 + 2} \\ &= \frac{2\left(\frac{3}{2}\right) - 1}{\frac{3}{2} + 2} \\ &= \frac{4}{7} \end{aligned}$$

$$\begin{aligned} r_3 = f(r_2) &= \frac{2\left(\frac{4}{7}\right) - 1}{\frac{4}{7} + 2} \\ &= \frac{1}{18} \end{aligned}$$

$$\begin{aligned} r_4 = f(r_3) &= \frac{2\left(\frac{1}{18}\right) - 1}{\frac{1}{18} + 2} \\ &= -\frac{16}{37} \end{aligned}$$

(ii) Observe that $f(r)$ is only undefined when $r = -2$, so to choose r_1 so that $r_3 = f(r_2)$ is undefined, we need $r_2 = -2$. Therefore, $r_2 = f(r_1) = -2$, which gives rise to the equation $\dfrac{2r_1 - 1}{r_1 + 2} = -2$. Multiplying both sides by $r_1 + 2$ gives $2r_1 - 1 = -2(r_1 + 2)$, which can be rearranged to get $4r_1 = -3$, so $r_1 = -\frac{3}{4}$. Indeed, if $r_1 = -\frac{3}{4}$, then $r_2 = -2$ and r_3 is undefined.

(b) (i)

$$\begin{aligned} r_2 = f(r_1) &= \frac{\frac{3}{7} + 3}{2\left(\frac{3}{7}\right) - 1} \\ &= -24. \end{aligned}$$

$$\begin{aligned} r_3 = f(r_2) &= \frac{-24 + 3}{2(-24) - 1} \\ &= \frac{-21}{-49} \\ &= \frac{3}{7} \\ &= r_1 \end{aligned}$$

We have that $r_3 = r_1$, so this means $r_4 = f(r_3) = f(r_1) = r_2$, and using these facts, $r_5 = f(r_4) = f(r_2) = r_3 = r_1$. This will continue to get that $r_n = \frac{3}{7}$ when n is odd and $r_n = -24$ when n is even.

(ii) First observe that $f(r)$ is undefined only when $2r-1=0$, or $r=\frac{1}{2}$. Therefore, if $r_1=\frac{1}{2}$, then $r_2=f(r_1)$ is undefined.

Now suppose $f(r)=\frac{1}{2}$. Then $\dfrac{r+3}{2r-1}=\frac{1}{2}$, which can be rearranged to get the equation $2r+6=2r-1$. This equation implies $6=-1$, which is nonsense, and so we conclude that there is no r such that $f(r)=\frac{1}{2}$.

The only way that the sequence can fail to be defined somewhere is if $r_n=\frac{1}{2}$ for some n. This happens if $r_1=\frac{1}{2}$, but since $\frac{1}{2}$ is never the output of f, it is not possible that $r_n=\frac{1}{2}$ unless $n=1$. Therefore, $r_1=\frac{1}{2}$ is the only starting value for which the sequence is undefined somewhere.

Assuming $r\neq\frac{1}{2}$, we will now compute a general expression for $f(f(r))$. Since $r\neq\frac{1}{2}$ and $f(r)\neq\frac{1}{2}$ regardless of r, there will be no issues with the expression being undefined.

$$\begin{aligned}
f(f(r)) &= \frac{\frac{r+3}{2r-1}+3}{2\left(\frac{r+3}{2r-1}\right)-1} \\
&= \frac{r+3+3(2r-1)}{2(r+3)-(2r-1)} && \text{(multiply through by } 2r-1\text{)} \\
&= \frac{7r}{7} \\
&= r
\end{aligned}$$

and so we have that $f(f(r))=r$ for all $r\neq\frac{1}{2}$. We can use this equation to get $r_3=f(r_2)=f(f(r_1))=r_1$. Next, we can compute $r_5=f(r_4)=f(f(r_3))=r_3=r_1$. Continuing, we see that $r_n=r_1$ for all odd n. Similarly, $r_4=f(r_3)=f(f(r_2))=r_2$, and we can continue with this reasoning to get that $r_n=f(r_1)=r_2=\dfrac{r_1+3}{2r_1-1}$ for all even n.

To answer the given question, since 2023 is odd, $r_{2023}=r_1$ and since 2024 is even, $r_{2024}=\dfrac{r_1+3}{2r_1-1}$.

(c) (i) The values along with their decimal approximations are in the table below.

r_1	1	1
r_2	$\frac{3}{2}$	1.5
r_3	$\frac{7}{5}$	1.4
r_4	$\frac{17}{12}$	1.416667
r_5	$\frac{41}{29}$	1.413793
r_6	$\frac{99}{70}$	1.414286
r_7	$\frac{239}{169}$	1.414201
r_8	$\frac{577}{408}$	1.414216
r_9	$\frac{1393}{985}$	1.414213

(ii) We will work with the quantity $\dfrac{f(r)-\sqrt{2}}{r-\sqrt{2}}$ and worry about the absolute value later.

$$\begin{aligned}
\frac{f(r)-\sqrt{2}}{r-\sqrt{2}} &= \frac{\frac{r+2}{r+1}-\sqrt{2}}{r-\sqrt{2}} \\
&= \frac{r+2-\sqrt{2}(r+1)}{(r+1)(r-\sqrt{2})} && \text{(multiply through by } r+1\text{)} \\
&= \frac{r-\sqrt{2}+2-\sqrt{2}r}{(r+1)(r-\sqrt{2})} \\
&= \frac{r-\sqrt{2}-\sqrt{2}(r-\sqrt{2})}{(r+1)(r-\sqrt{2})} \\
&= \frac{(r-\sqrt{2})(1-\sqrt{2})}{(r+1)(r-\sqrt{2})} \\
&= \frac{1-\sqrt{2}}{r+1} && \text{(cancel } r-\sqrt{2}\text{)}
\end{aligned}$$

The result now follows by taking the absolute value of both sides.

(iii) Observe that if r is positive, then $f(r)=\dfrac{r+2}{r+1}$ is also positive since $r+2$ and $r+1$ are both positive. It follows that if r_1 is positive, then r_n is positive for all n. Since $r_n>0$ for all n, $r_n+1>1$ for all n, and taking reciprocals, we get $\dfrac{1}{r_n+1}<1$ for all n. Since r_n+1 is positive, $\left|\dfrac{1}{r_n+1}\right|=\dfrac{1}{r_n+1}$, so we actually get that $\left|\dfrac{1}{r_n+1}\right|<1$ for all n.

We will now show that $|1-\sqrt{2}|<\frac{1}{2}$ (you may already believe this to be true, but the proof presented does not assume that we have a known approximation of $\sqrt{2}$). To see this, observe that $8<9$, and so $\sqrt{8}<\sqrt{9}$ which is the same as $2\sqrt{2}<3$. Dividing both sides by 2, we get $\sqrt{2}<\frac{3}{2}$. Subtracting $\frac{1}{2}+\sqrt{2}$ from both sides gives $-\frac{1}{2}<1-\sqrt{2}$. Now observe that $1<2$, so $\sqrt{1}<\sqrt{2}$ or $1<\sqrt{2}$. Therefore, $1-\sqrt{2}<0$.

We have shown that $-\frac{1}{2}<1-\sqrt{2}<0$, which implies that $|1-\sqrt{2}|<\frac{1}{2}$. Combining this with $\left|\dfrac{1}{r_n+1}\right|<1$, we get

$$\left|\frac{1-\sqrt{2}}{r_n+1}\right|=\left|\frac{1}{r_n+1}\right||1-\sqrt{2}|<1\times\frac{1}{2}=\frac{1}{2}$$

so $\left|\dfrac{1-\sqrt{2}}{r_n+1}\right|<\frac{1}{2}$.

Since $r_n=f(r_{n-1})$ for all $n\geq 2$, we can apply part (ii) to get

$$\left|\frac{r_n-\sqrt{2}}{r_{n-1}-\sqrt{2}}\right|=\left|\frac{f(r_{n-1})-\sqrt{2}}{r_{n-1}-\sqrt{2}}\right|=\left|\frac{1-\sqrt{2}}{r_{n-1}+1}\right|<\frac{1}{2}$$

where the final inequality comes from applying what we showed above.

This implies that for all $n \geq 2$ we have

$$|r_n - \sqrt{2}| < \frac{1}{2}|r_{n-1} - \sqrt{2}| \qquad (*)$$

Now let's return to the inequality in the question, which is $|r_n - \sqrt{2}| < \dfrac{1}{2^{n-1}}|r_1 - \sqrt{2}|$.

When $n = 2$, this inequality is $|r_2 - \sqrt{2}| < \frac{1}{2}|r_1 - \sqrt{2}|$, which is exactly $(*)$ when $n = 2$. We have already shown that $(*)$ is true for all n, so this means the desired inequality is true for $n = 2$.

When $n = 3$, we can apply $(*)$ to get $|r_3 - \sqrt{2}| < \frac{1}{2}|r_2 - \sqrt{2}|$, but we have just shown that $|r_2 - \sqrt{2}| < \frac{1}{2}|r_1 - \sqrt{2}|$. Therefore,

$$|r_3 - \sqrt{2}| < \frac{1}{2}|r_2 - \sqrt{2}| < \frac{1}{2}\left(\frac{1}{2}|r_1 - \sqrt{2}|\right) = \frac{1}{2^2}|r_1 - \sqrt{2}|$$

which shows that the desired inequality holds for $n = 3$.

By similar reasoning, we can use the fact that the inequality holds for $n = 3$ to prove that it holds for $n = 4$, then we can use that it holds for $n = 4$ to prove that it holds for $n = 5$, and so on to show that the inequality holds for all positive integers $n \geq 2$. We can formalize this using *mathematical induction*.

Assume that $k \geq 2$ is an integer for which the inequality $|r_k - \sqrt{2}| < \dfrac{1}{2^{k-1}}|r_1 - \sqrt{2}|$ is true. Using $(*)$ with $n = k + 1$, we have the following

$$|r_{k+1} - \sqrt{2}| < \frac{1}{2}|r_k - \sqrt{2}|$$

and now using the *inductive hypothesis*, that $|r_k - \sqrt{2}| < \frac{1}{2^{k-1}}|r_1 - \sqrt{2}|$, we get

$$|r_{k+1} - \sqrt{2}| < \frac{1}{2}\left(|r_k - \sqrt{2}|\right) < \frac{1}{2}\left(\frac{1}{2^{k-1}}|r_1 - \sqrt{2}|\right) = \frac{1}{2^k}|r_1 - \sqrt{2}|$$

but $k = (k + 1) - 1$, so we have shown that the inequality holds for the integer $k + 1$.

To summarize, we have shown that the inequality holds for $n = 2$, and we have shown that if the inequality holds for an integer, then it holds for the next integer. This shows that the inequality holds for all integers $n \geq 2$.

Finally, since $|r_1 - \sqrt{2}|$ is a fixed quantity, the quantity $\dfrac{1}{2^{n-1}}|r_1 - \sqrt{2}|$ must get closer and closer to 0 as n gets larger and larger. We also have that $|r_n - \sqrt{2}| < \dfrac{1}{2^{n-1}}|r_1 - \sqrt{2}|$ for all n, which means the quantity $|r_n - \sqrt{2}|$, which is positive, is also getting closer and closer to 0 as n gets larger and larger. It follows that $r_n - \sqrt{2}$ is very close to zero for very large n, which means r_n is very close to $\sqrt{2}$ for very large n. Keep in mind that this is true for *any* positive starting value r_1.

(d) Here is a brief summary of the behaviours of the sequences.

When $f(r) = \dfrac{r-3}{r-2}$, the sequence will repeat with period 3 as long as at least three terms in the sequence are defined. The only values of r_1 for which the sequence has an undefined term are $r_1 = 2$ and $r_1 = 1$.

When $f(r) = \dfrac{r-1}{5r+3}$, the sequence will repeat with period 4 as long as at least four terms are defined. The only values of r_1 for which the sequence has an undefined term are $r_1 = -\frac{3}{5}$, $r_1 = -\frac{1}{5}$, and $r_1 = \frac{1}{5}$.

When $f(r) = \dfrac{r-1}{r+2}$, the sequence will repeat with period 6 as long as at least six terms are defined. The only values of r_1 for which the sequence has an undefined term are $r_1 = -2$, $r_1 = -1$, $r_1 = -\frac{1}{2}$, $r_1 = 0$, and $r_1 = 1$.

When $f(r) = \dfrac{2r+2}{3r+3}$, the sequence is undefined after r_1 if $r_1 = -1$. Otherwise, the sequence has $r_n = \frac{2}{3}$ for all $n \geq 2$, regardless of the value of r_1.

Note: These examples, together with the one in part (b), show that the sequences can be periodic with period 1, 2, 3, 4, or 6. Do you think that any other periods are possible?

When $f(r) = \dfrac{r+1}{r-2}$, the sequence converges to $\dfrac{3-\sqrt{13}}{2}$ unless the sequence has an undefined term. To get an idea of why, try solving the equation $\dfrac{r+1}{r-2} = r$ for r (this can be rearranged to a quadratic equation in r). There are infinitely many values of r_1 for which r_n is undefined. The first few of them are

$$2, 5, \frac{11}{4}, \frac{26}{7}, \frac{59}{19}, \frac{137}{40}, \frac{314}{97}, \frac{725}{217}, \ldots$$

Can you determine how these values are calculated? You might be interested in computing decimal approximations of these values and looking for a pattern.

As a final remark, we note that not all sequences are "well-behaved" (either periodic or approaching some value). For an example of a more chaotic sequence, try exploring the example in part (a) a little further. Can you see any pattern at all?